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Estimation of Logistic Parameters Using a Fuzzy Least-squares Method and Different Types of Moments

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Authors' contributions

This work was carried out in collaboration between all authors. Author HMZ designed the study, wrote the protocol, and wrote the first draft of the manuscript. Author AAES managed the literature searches, analyses of the study performed the spectroscopy analysis and author NATAEM managed the experimental process and identified the species of plant. All authors read and approved the final manuscript.

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ABSTRACT

The main attention of this paper is to deduce the estimators of the parameters of the Logistic distribution using five estimating methods, namely, the fuzzy least-squares method, the LQ-moments (linear quantile moments) with three cases (trimean, median and Gastwirth), TL-moments (trimmed linear moments) with different individual cases, L-moments (linear moments) and the maximum likelihood method. Also, a comparison between the performances of these estimators using simulations is given. According to these comparisons, it is shown that the proposed fuzzy least-squares algorithm is preferred for large sample size.

Keywords: Logistic distribution; fuzzy least-squares; maximum likelihood; TL-moments; L-moments; LL-moments; LH-moments; LQ-moments; simulations.

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1. INTRODUCTION

The logistic distribution is largely used to model events that happen in several fields such as medicine, social and natural sciences. Also, the logistic distribution arises frequently in statistical modelling. It is mostly used in regression analysis and studies on population growths. It is utilized in the study of survival data, graduation of mortality statistics and is used in several applications as a substitute for the normal distribution. The cumulative distribution function of the two-parameter logistic distribution is defined by the following:

$$F(x;\alpha,\beta)=\frac{1}{1+e^{-\frac{(x-\alpha)}{\beta}}}, \qquad -\infty<x<\infty \quad (1\text{-}1)$$

and the probability density function is:

$$f(x;\alpha,\beta)=\frac{e^{-\frac{(x-\alpha)}{\beta}}}{\beta\left(1+e^{-\frac{(x-\alpha)}{\beta}}\right)^{2}}, \qquad -\infty<x<\infty \quad (1\text{-}2)$$

where $-\infty<\alpha<\infty$ and $\beta>0$ refer to the location and scale parameters. For more information on the logistic distribution see chapter 23 of Johnson et al. [1]. The corresponding quantile function of the logistic distribution is given by:

$$Q(u)=\alpha+\beta\ln\left(\frac{u}{1-u}\right), \qquad 0<u<1 \quad (1\text{-}3)$$

The distribution is symmetric about the location parameter α and has the same value for the mode, the median and the mean $=\alpha$, while the variance is given by:

$$Var(X)=\frac{\beta^2\pi^2}{3}, \qquad ,\beta>0 \quad (1\text{-}4)$$

Hung and Liu [2] apply a robust fuzzy least-squares method for estimating the parameters of the Weibull distribution in the presence of outliers. To tackle this problem, a cluster-wise fuzzy least-squares algorithm in the presence of a noise cluster is proposed. Numeric comparisons between the fuzzy least-squares algorithm and other methods (the least absolute deviation, the least-squares, the weighted least-squares and Drapella and Kosznik [3]) are executed. These comparisons show that the estimates of the location and scale parameters of the proposed fuzzy least-squares algorithm is preferred for large sample size.

Hosking [4] proposed the L-moments concept and deduced that the mean of the distribution should be finite to have meaningful L-moments of the probability distribution. Also the condition for having finite standard errors of L-moments of the distribution is that the distribution has a finite variance and L-moments, also linear functions forms of the data, will be less sensitive than are conventional moments to sampling variability of the outliers in the data. Elamir and Seheult [5] defined the TL-moments and reached at the conclusion that TL-moments are more robust to outliers, TL-Moments give zero values as weights to the extreme observations, they are simple to compute and a population TL-Moments can be well defined in the case when the relative population L-Moments (or central moment) does not exist.

Mudholkar and Hutson [6] defined the concept of the LQ-moments and arrived at the conclusion that LQ-moments are usually simpler to compute and estimate than L-moments, LQ-moments usually exist and unique and their limiting distributions are simpler to obtain. Abu El-Magd [7] found the TL-moments and LQ-moments estimators of the exponentiated generalized extreme value distribution. She gave a numeric simulation comparison between TL-moments estimators with other estimation techniques (L-moments estimators, LQ-moment estimators and the method of moment estimators) focusing on their biases and root mean squared errors.

Zaher et al. [8] gave the estimation for the two-parameter Pareto distribution using the method of fuzzy least-squares. Also, they gave a comparison between the fuzzy least-squares estimator with other types of estimators. Firstly, they obtained the LQ-moments, TL-moments and L-moments formulas for the two-parameter Pareto distribution. Secondly, they obtained the LQ-moments estimator, TL-moments estimator and L-moments estimator for the Pareto distribution. Finally, numeric comparisons implemented between the proposed method and the available methods. Due to these comparisons, they arrived at the conclusion that the proposed fuzzy least-squares estimator is preferred all times.

The attention of this paper is to give the fuzzy least-squares method to find the estimates of the logistic distribution parameters and to give the LQ-moments and TL-moments of the logistic distribution. The fuzzy least-squares estimators (FLSEs), maximum likelihood estimators (MLEs), L-moment estimators (LMEs), TL-moment estimators (TLMEs) and LQ-moment estimators (LQMEs) of the logistic distribution will be derived. A numeric simulation is introduced to compare these methods of estimation focusing on their root mean squared errors (RMSEs) and their biases that will be obtained.

The remaining sections are as follows. In the section two, the maximum likelihood estimators (MLEs) will be obtained for the logistic distribution. In section three, the L-moments and the TL-moments with several special cases for the logistic distribution will be derived. Also, the L-moments estimators (LMEs) and TL-moment estimators (TLMEs) will be given for the same distribution. In section four, the LQ-moments with different special cases (trimean, median and Gastwirth) of the logistic distribution will be obtained and from these moments the LQ-moment estimators (LQMEs) for the three cases (trimean, median and Gastwirth) will be derived for the same distribution. In section five, the fuzzy least-squares estimators (FLSEs) of the logistic distribution will be obtained by using the fuzzy least-squares method. In section six, a numeric simulation study to compare the properties of the LMEs, TLMEs, LQMEs, MLEs and the FLSEs of the logistic distribution will be obtained. Finally, the results and conclusion of this comparison between different estimators for the logistic distribution will be given.

2. MAXIMUM LIKELIHOOD ESTIMATORS

The likelihood function, *l*, for a sample $(x_1, x_2, ..., x_n)$ of the logistic distribution has the following form:

$$l(\alpha,\beta)=\prod_{i=1}^{n}\frac{e^{-\frac{(x_i-\alpha)}{\beta}}}{\beta\left(1+e^{-\frac{(x_i-\alpha)}{\beta}}\right)^2}=\frac{e^{-\sum_{i=1}^{n}\frac{(x_i-\alpha)}{\beta}}}{\beta^n\left(\prod_{i=1}^{n}\left(1+e^{-\frac{(x_i-\alpha)}{\beta}}\right)\right)^2} \quad (2\text{-}1)$$

and taking logarithms,

$$L=\log l(\alpha,\beta)=-\sum_{i=1}^{n}\frac{(x_i-\alpha)}{\beta}-n\log\beta-2\sum_{i=1}^{n}\log\left(1+e^{-\frac{(x_i-\alpha)}{\beta}}\right) \quad (2\text{-}2)$$

Hence

$$\frac{\partial L}{\partial\alpha}=\frac{n}{\beta}-\frac{2}{\beta}\sum_{i=1}^{n}\frac{e^{\frac{-(x_i-\alpha)}{\beta}}}{\left(1+e^{\frac{-(x_i-\alpha)}{\beta}}\right)} \quad (2\text{-}3)$$

And

$$\frac{\partial L}{\partial\beta}=\sum_{i=1}^{n}\frac{(x_i-\alpha)}{\beta^2}-\frac{n}{\beta}-\frac{2}{\beta^2}\sum_{i=1}^{n}\frac{e^{\frac{-(x_i-\alpha)}{\beta}}}{\left(1+e^{\frac{-(x_i-\alpha)}{\beta}}\right)}(x_i-\alpha) \quad (2\text{-}4)$$

The estimators of maximum likelihood for α and β are the result of solving the system of equations obtained from the partial derivative of the log-likelihood function, that is,

$$\frac{n}{\beta^*}-\frac{2}{\beta^*}\sum_{i=1}^{n}\frac{e^{-\frac{(x_i-\alpha^*)}{\beta^*}}}{\left(1+e^{-\frac{(x_i-\alpha^*)}{\beta^*}}\right)}=0 \tag{2-5}$$

and

$$\sum_{i=1}^{n}\frac{(x_i-\alpha^*)}{\beta^{*2}}-\frac{n}{\beta^*}-\frac{2}{\beta^{*2}}\sum_{i=1}^{n}\frac{e^{-\frac{(x_i-\alpha^*)}{\beta^*}}}{\left(1+e^{-\frac{(x_i-\alpha^*)}{\beta^*}}\right)}(x_i-\alpha^*)=0 \tag{2-6}$$

Simplifying, we get the equations from which the estimates can be found numerically

$$\sum_{i=1}^{n}\frac{1}{\left(1+e^{\frac{(x_i-\alpha^*)}{\beta^*}}\right)}=\frac{n}{2} \tag{2-7}$$

and

$$\sum_{i=1}^{n}\frac{(x_i-\alpha^*)}{\beta^*}\frac{\left(1-e^{-\frac{(x_i-\alpha^*)}{\beta^*}}\right)}{\left(1+e^{-\frac{(x_i-\alpha^*)}{\beta^*}}\right)}=n \tag{2-8}$$

3. TL-MOMENTS ESTIMATORS AND L-MOMENTS ESTIMATORS

In this section, the TL-moments of the logistic distribution will be obtained. From the TL-moments with generalized trimmed, several special cases can be found as the TL-moments with the first trimmed, LH-moments, L-moments and LL-moments for the logistic distribution.

3.1 TL-moments

Let, $X_1, X_2, \ldots, X_n$ be a conceptual random sample of n observations from a continuous distribution and let, $X_{(1:n)} \le X_{(2:n)} \le \ldots \le X_{(n:n)}$ denote the corresponding order statistics. Elamir and Seheult [5] defined the r[th] TL-moment $\lambda_r^{(s,t)}$ as follows:

$$\lambda_r^{(s,t)}=\frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}E\left(X_{(r+s-k:r+s+t)}\right), \qquad r=1,2,\ldots \tag{3-1}$$

The TL-moments reduce to L-moments (Hosking [4]) when s = t = 0. They considered the symmetric case ($s=t$). Hosking [9] have got several conclusions for the TL-moments with common trimmed for s and t (symmetric case ($s = t$) and asymmetric case ($s \neq t$)). They also obtained the coefficient of variation of TL-moments TL-CV, the TL-skewness and the TL-kurtosis as follows:

$$\tau^{(s,t)} = \lambda_2^{(s,t)} \big/ \lambda_1^{(s,t)}, \ \tau_3^{(s,t)} = \lambda_3^{(s,t)} \big/ \lambda_2^{(s,t)}, \ and \ \tau_4^{(s,t)} = \lambda_4^{(s,t)} \big/ \lambda_2^{(s,t)}. \tag{3-2}$$

Maillet andMédecin[10] gave the relation between the first TL-moments and the rth TL-moments with generalized trimmed for s and t (symmetric case $(s = t)$ and asymmetric case $(s \neq t)$). Actually, it is sufficient to compute TL-moments of order one to get all TL-moments. They have got the following rth TL-moments:

$$\lambda_r^{(s,t)} = \frac{1}{r}\sum_{j=0}^{r-1}(-1)^j \binom{r-1}{j} \lambda_1^{(r+s-j-1,t+j)}, \quad r = 1,\ 2,\ ,3,\ ... \tag{3-3}$$

where t, s = 0, 1, 2,… . This relation had a relative importance and helped to make easier calculations for the rth TL-moments with any trimmed and L-moments as special cases of the rth TL-moments with common trimmed for t and s. They showed that the TL-moments approach is a general framework that encompass the LH-moments, LL-moments and the L-moments. Here, we obtain the rth TL-moments for the logistic distribution as follows:

$$\begin{aligned}\lambda_r^{(s,t)} = \frac{1}{r}\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}\Bigg[\alpha - \frac{(r+s+t)!\beta}{(r+s-j-1)!(t+j)!}\Bigg(\sum_{k=0}^{t+j}(-1)^k\binom{t+j}{k}(r+s-j+k)^{-2} \\ - \sum_{k=0}^{r+s-j-1}(-1)^k\binom{r+s-j-1}{k}(t+j+k+1)^{-2}\Bigg)\Bigg], \qquad r = 1,\ 2,\ ...,\end{aligned} \tag{3-4}$$

Due to the given relations, the first four TL-moments with common trimmed for t and s (t, s = 0, 1, 2, …,) of the logistic distribution will be:

$$\lambda_1^{(s,t)} = \alpha - \frac{(s+t+1)!\beta}{(s)!(t)!}\Bigg[\sum_{k=0}^{t}(-1)^k\binom{t}{k}(s+k+1)^{-2} - \sum_{k=0}^{s}(-1)^k\binom{s}{k}(t+k+1)^{-2}\Bigg], \tag{3-5}$$

$$\begin{aligned}\lambda_2^{(s,t)} = \frac{\beta}{2}\frac{(s+t+2)!}{(s+1)!(t+1)!}\Bigg[(t+1)^{-1} + (s+1)^{-1} - \sum_{k=0}^{t}(-1)^k\binom{t+1}{k+1}(s+k+2)^{-1} \\ - \sum_{k=0}^{s}(-1)^k\binom{s+1}{k+1}(t+k+2)^{-1}\Bigg],\end{aligned} \tag{3-6}$$

$$\begin{aligned}\lambda_3^{(s,t)} = \frac{\beta}{3}\frac{(s+t+3)!}{(s+2)!(t+2)!}\Big[(t+2)\big((t+1)^{-1} + (s+2)^{-1}(s+3)\big) - (s+2)\big((s+1)^{-1} + (t+2)^{-1}(t+3)\big) \\ - \sum_{k=0}^{t}(-1)^k\binom{t+2}{k+2}(s+k+3)^{-1}(s+k+4) + \sum_{k=0}^{s}(-1)^k\binom{s+2}{k+2}(t+k+3)^{-1}(t+k+4)\Big],\end{aligned} \tag{3-7}$$

and

$$\lambda_4^{(s,t)} = \frac{\beta}{4}\frac{(s+t+4)!}{(s+3)!(t+3)!}\Big[(t+3)(t+2)(t+1)^{-1} + (s+3)(s+2)(s+1)^{-1}$$
$$- (t+3)(s+3)\left((t+4)(t+2)^{-1} + (s+4)(s+2)^{-1}\right) + (s+3)(s+2)(t+3)^{-1}\left((t+4)(t+2)+3\right)$$
$$+ (t+3)(t+2)(s+3)^{-1}\left((s+4)(s+2)+3\right) - \sum_{k=0}^{t}(-1)^k\binom{t+3}{k+3}(s+k+4)^{-1}(s+k+5)(s+k+6)$$
$$- \sum_{k=0}^{s}(-1)^k\binom{s+3}{k+3}(t+k+4)^{-1}(t+k+5)(t+k+6)\Big]. \qquad (3\text{-}8)$$

From these results, we can find the TL- coefficient of variation $\tau^{(s,t)}$, TL-skewness $\tau_3^{(s,t)}$ and TL-kurtosis $\tau_4^{(s,t)}$ with common cut for t and s (t, s = 0, 1, 2, …,) for the logistic distribution.

3.2 Special Cases

Many special cases will be obtained from the first four TL-Moments with generalized trimmed for the logistic distribution such as the TL-moments with the first cut, LH-moments, LL-moments and L-moments for the logistic distribution.

3.2.1 TL-moments with the first trimmed ($s = t = 1$):

By substituting $s = 1$ and $t = 1$ in equations (3-5), (3-6), (3-7) and (3-8), the first four TL-moments with the first trimmed for the logistic distribution will be:

$$\lambda_1^{(1)} = \alpha, \qquad (3\text{-}9)$$

$$\lambda_2^{(1)} = \frac{\beta}{2}, \qquad (3\text{-}10)$$

$$\lambda_3^{(1)} = Zero, \qquad (3\text{-}11)$$

and

$$\lambda_4^{(1)} = \frac{\beta}{24}. \qquad (3\text{-}12)$$

Based on these results, the TL- coefficient of variation $\tau^{(1,1)}$, TL-kurtosis $\tau_4^{(1,1)}$ and TL-skewness $\tau_3^{(1,1)}$ can be obtained with the first trimmed for the logistic distribution. The results for the first four TL-moments with the first trimmed for the logistic distribution as a special case of the TL-moments with common trimmed for t and s (t, s = 0, 1, 2, …,) of the logistic distribution are the same results of Elamir and Seheult [5] for the logistic distribution.

3.2.2 L-moments ($s = t = 0$):

By substituting $s = 0$, and $t = 0$ in the rth TL-moments for the logistic distribution, we can obtain the rth L-moments for the logistic distribution as follows:

$$\lambda_r = \frac{1}{r}\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}\left[\alpha - \frac{(r)!\beta}{(r-j-1)!(j)!}\left(\sum_{k=0}^{j}(-1)^k\binom{j}{k}(r+k-j)^{-2}\right.\right.$$
$$\left.\left. - \sum_{k=0}^{r-j-1}(-1)^k\binom{r-j-1}{k}(k+j+1)^{-2}\right)\right], \qquad r = 1, 2, ..., \qquad (3\text{-}13)$$

Also, we can obtain the first four L-moments for the logistic distribution by substituting $s = 0$, and $t = 0$ in equations (3-5), (3-6), (3-7) and (3-8), as a particular case from the TL-moments for the logistic distribution. The first four L-moments for the logistic distribution will be:

$$\lambda_1 = \alpha, \tag{3-14}$$

$$\lambda_2 = \beta, \tag{3-15}$$

$$\lambda_3 = Zero, \tag{3-16}$$

and

$$\lambda_4 = \frac{\beta}{6}, \tag{3-17}$$

and with the first four L-moments, we can get the L-coefficient of variation $\tau = \lambda_2 / \lambda_1$, L-kurtosis $\tau_4 = \lambda_4 / \lambda_2$ and L-skewness $\tau_3 = \lambda_3 / \lambda_2$ for the logistic distribution. The results for the first four L-moments for the logistic distribution as a particular case of the TL-moments with common trimmed for t and s (t, s = 0, 1, 2, …,) of the logistic distribution are the same results of Hosking [4] for the logistic distribution.

3.2.3 LH-moments ($t = 0$):

The LH-moments are linear functions of the expectations of the highest rank statistic and were given by Wang [11] as an adapted version of L-moments, to typify the upper part of the distribution. When it is required to focus on extreme events, the LH-moment method allows to give more importance to the largest items. When $s = 0$, the LH-moment corresponds with the L-moments. As s increases, LH-moments give more attention on the characteristics of the upper portion of the data. Wang [11] arrived at the conclusion that the method of LH-moments gave large sampling variability for high s and advised not to use values more than 4.

By substituting $t = 0$, in equations (3-5), (3-6), (3-7) and (3-8), the first four LH-moments with generalized trimmed for s of the logistic distribution will be:

$$\lambda_1^{(s,0)} = \alpha - (s+1)\beta\left[(s+1)^{-2} - \sum_{k=0}^{s} (-1)^k \binom{s}{k} (k+1)^{-2} \right], \tag{3-18}$$

$$\lambda_2^{(s,0)} = \frac{\beta}{2}(s+2)\left[1 + (s+1)^{-1} - (s+2)^{-1} - \sum_{k=0}^{s} (-1)^k \binom{s+1}{k+1} (k+2)^{-1} \right], \tag{3-19}$$

$$\begin{aligned} \lambda_3^{(s,0)} = \frac{\beta}{6}(s+3)\Bigg[& 2 + 2(s+2)^{-1}(s+3) - (s+2)\left((s+1)^{-1} + \frac{3}{2} \right) - (s+3)^{-1}(s+4) \\ & + \sum_{k=0}^{s} (-1)^k \binom{s+2}{k+2} (k+3)^{-1}(k+4) \Bigg], \end{aligned} \tag{3-20}$$

and

$$\lambda_4^{(s,0)} = \frac{\beta}{24}(s+4)\Big[6+(s+3)(s+2)(s+1)^{-1}-3(s+3)\big(2+(s+4)(s+2)^{-1}\big)+11/3(s+3)(s+2)$$
$$+6(s+3)^{-1}\big((s+4)(s+2)+3\big)-(s+4)^{-1}(s+5)(s+6)-\sum_{k=0}^{s}(-1)^k\binom{s+3}{k+3}(k+4)^{-1}(k+5)(k+6)\Big]. \quad \text{(3-21)}$$

From these results we can get the LH-coefficient of variation $\tau^{(s,0)} = \lambda_2^{(s,0)}/\lambda_1^{(s,0)}$, LH-kurtosis $\tau_4^{(s,0)} = \lambda_4^{(s,0)}/\lambda_2^{(s,0)}$ and LH-skewness $\tau_3^{(s,0)} = \lambda_3^{(s,0)}/\lambda_2^{(s,0)}$ with generalized trimmed for s of the logistic distribution. Furthermore, for s = 1, 2, 3, …, the LH-moments can be found with any cut s for the logistic distribution.

3.2.4 LL-moments ($s=0$):

The LL-moments are linear relations of the expectations of the lowest rank statistic and were given by Bayazit and Önöz [12]. L-moments are a particular case for $t=0$, and if t increases the importance of the lower portion of the data will be increased. By substituting $s=0$ in equations (3-5), (3-6), (3-7) and (3-8), the first four LL-moments with generalized trimmed for t can be obtained for the logistic distribution as follows:

$$\lambda_1^{(0,t)} = \alpha-(t+1)\beta\left[\sum_{k=0}^{t}(-1)^k\binom{t}{k}(k+1)^{-2}-(t+1)^{-2}\right], \quad \text{(3-22)}$$

$$\lambda_2^{(0,t)} = \frac{\beta}{2}(t+2)\left[1+(t+1)^{-1}-(t+2)^{-1}-\sum_{k=0}^{t}(-1)^k\binom{t+1}{k+1}(k+2)^{-1}\right], \quad \text{(3-23)}$$

$$\lambda_3^{(0,t)} = \frac{\beta}{6}(t+3)\Big[(t+2)\big((t+1)^{-1}+3/2\big)-2-2(t+2)^{-1}(t+3)+(t+3)^{-1}(t+4)$$
$$-\sum_{k=0}^{t}(-1)^k\binom{t+2}{k+2}(k+3)^{-1}(k+4)\Big], \quad \text{(3-24)}$$

and

$$\lambda_4^{(0,t)} = \frac{\beta}{24}(t+4)\Big[6+(t+3)(t+2)(t+1)^{-1}-3(t+3)\big((t+4)(t+2)^{-1}+2\big)$$
$$+6(t+3)^{-1}\big((t+4)(t+2)+3\big)+\frac{11}{3}(t+3)(t+2)$$
$$-(t+4)^{-1}(t+5)(t+6)-\sum_{k=0}^{t}(-1)^k\binom{t+3}{k+3}(k+4)^{-1}(k+5)(k+6)\Big]. \quad \text{(3-25)}$$

From these results, it is possible to get the LL-coefficient of variation $\tau^{(0,t)} = \lambda_2^{(0,t)}/\lambda_1^{(0,t)}$, LL-kurtosis $\tau_4^{(0,t)} = \lambda_4^{(0,t)}/\lambda_2^{(0,t)}$ and LL-skewness $\tau_3^{(0,t)} = \lambda_3^{(0,t)}/\lambda_2^{(0,t)}$ with generalized trimmed for t of the logistic distribution. Also, it is possible to get the LL-moments for the logistic distribution with any trimmed t for t = 1, 2, 3,... .

3.3 TL-moments Estimators

The TL-moment estimators (TLMEs) of the logistic distribution for the unknown parameters can be found using the equations of the first two population TL-moments ($\lambda_1^{(s,t)}, \lambda_2^{(s,t)}$) and the corresponding to the first two sample TL-moments ($l_1^{(s,t)}, l_2^{(s,t)}$) for the logistic distribution. Hosking [9] has got the first two TL-moments of the sample to be:

$$l_1^{(s,t)} = \frac{1}{\binom{n}{s+t+1}} \sum_{j=s+1}^{n-t} \binom{j-1}{s}\binom{n-j}{t} x_{(j:n)}, \tag{3-26}$$

and

$$l_2^{(s,t)} = \frac{1}{2\binom{n}{s+t+2}} \sum_{j=s+1}^{n-t} \binom{j-1}{s}\binom{n-j}{t}\left(\frac{(j-s-1)}{(s+1)} - \frac{(n-j-t)}{(t+1)}\right) x_{(j:n)} \tag{3-27}$$

It is clear that for the special case s = t = 0, the sample TL-moments are the same as the sample L-moments. Also, the TL-moment estimators (TLMEs) $\hat{\alpha}$ and $\hat{\beta}$ of the logistic distribution can be found using the following two equations:

$$l_1^{(s,t)} = \hat{\alpha} - \frac{(s+t+1)!\hat{\beta}}{(s)!(t)!}\left[\sum_{k=0}^{t}(-1)^k\binom{t}{k}(s+k+1)^{-2} - \sum_{k=0}^{s}(-1)^k\binom{s}{k}(t+k+1)^{-2}\right], \tag{3-28}$$

and

$$\begin{aligned} l_2^{(s,t)} = \frac{\hat{\beta}}{2}\frac{(s+t+2)!}{(s+1)!(t+1)!}\Bigg[&(t+1)^{-1} + (s+1)^{-1} - \sum_{k=0}^{t}(-1)^k\binom{t+1}{k+1}(s+k+2)^{-1} \\ &- \sum_{k=0}^{s}(-1)^k\binom{s+1}{k+1}(t+k+2)^{-1}\Bigg], \end{aligned} \tag{3-29}$$

which are valid for any trimmed t and s. These equations can be solved numerically for given values of t and s. This paper considers the special cases s = t = 1 and s = t = 0 for the logistic distribution.

3.4 L-moments Estimators

In this section, we will give the L-moment estimators (LMEs) for the logistic distribution. If refers to the order sample $X_{(1:n)} \le X_{(2:n)} \le \ldots \le X_{(n:n)}$, the first two sample L-moments will be given as follows:

$$l_1 = \frac{1}{n}\sum_{i=1}^{n} x_{(i:n)}, \tag{3-30}$$

and

$$l_2 = \frac{2}{n(n-1)}\sum_{i=1}^{n}(i-1)x_{(i:n)} - l_1. \tag{3-31}$$

Equating the first two L-moments of the population λ_1, λ_2, to a similar sample L-moments l_1, l_2, we will get:

$$l_1 = \alpha^{**}, \tag{3-32}$$

and

$$l_2 = \beta^{**} . \tag{3-33}$$

Then, the LMEs of α and β, say α^{**} and k^{**}, respectively, can be found by using numerical examples.

4. LQ-MOMENTS ESTIMATORS

In this section, the use of the LQ-moments for finding the unknown parameters of the logistic distribution will be derived. Three special cases (trimean, median and Gastwirth) of the LQ-moments used to find the unknown parameters of the logistic distribution.

4.1 LQ-moments

Let, $X_1, X_2, ..., X_n$ be a random sample from a continuous distribution function $F(x)$ with quantile function $Q_X(u) = F_X^{-1}(u)$, also, $X_{(1:n)} \le X_{(2:n)} \le ... \le X_{(n:n)}$ denote the order statistics. Mudholkar and Hutson [6] introduced the r[th] population LQ-moments ζ_r of X, as follows:

$$\zeta_r = r^{-1} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \tau_{p,d}(X_{(r-k:r)}), \qquad r = 1,2,... \tag{4-1}$$

where $0 \le d \le 1/2$, $\quad 0 \le p \le 1/2$, and

$$\tau_{p,d}(X_{(r-k:r)}) = pQ_{X_{(r-k:r)}}(d) + (1-2p)Q_{X_{(r-k:r)}}(1/2) + pQ_{X_{(r-k:r)}}(1-d). \tag{4-2}$$

The linear combination $\tau_{p,d}$ is a 'quick' measure of the location of the sampling distribution of the order statistic $X_{(r-k:r)}$. The candidates for $\tau_{p,d}$ including the function generating the common quick estimators by using the trimean ($p = 1/4,\ d = 1/4$), the median ($p = 0.5,\ d = 0.5$)and the Gastwirth ($p = 0.3,\ d = 1/3$). They also gave the LQ-kurtosis and LQ-skewness for the population by $\eta_4 = \zeta_4/\zeta_2$ and $\eta_3 = \zeta_3/\zeta_2$ respectively; it may be employed for describing the population and estimating its parameters. The symmetrical distributions have the value zero for The LQ-skewness.

The LQ-moments with the three cases (trimean, median, and Gastwirth) will be given for the logistic distribution as follows:

4.1.1 The trimean case ($p = 1/4,\ d = 1/4$):

Employing the quantile function for the logistic distribution, then the first four LQ-moments for the logistic distribution will be given as follows:

$$\xi_1 = \alpha + \frac{1}{4}\left[Q_\circ(0.25) + 2Q_\circ(0.5) + Q_\circ(0.75)\right], \tag{4-3}$$

$$\xi_2 = \frac{1}{8}\left[Q_\circ(0.866) + 2Q_\circ(0.707) - 2Q_\circ(0.293) - Q_\circ(0.134)\right], \tag{4-4}$$

$$\xi_3 = \frac{1}{12}\left[Q_\circ(0.909) + 2Q_\circ(0.794) - 2Q_\circ(0.674) + Q_\circ(0.630) - 4Q_\circ(0.5) + Q_\circ(0.370) - 2Q_\circ(0.326) + 2Q_\circ(0.206) + Q_\circ(0.091)\right], \tag{4-5}$$

and

$$\xi_4 = \frac{1}{16}\left[Q_\circ(0.931) + 2Q_\circ(0.841) - 3Q_\circ(0.757) + Q_\circ(0.707) - 6Q_\circ(0.614) + 3Q_\circ(0.544) - 3Q_\circ(0.456) + 6Q_\circ(0.386) - Q_\circ(0.293) + 3Q_\circ(0.243) - 2Q_\circ(0.159) - Q_\circ(0.069)\right]. \tag{4-6}$$

Where

$$Q_\circ(u) = \beta \ln\left(\frac{u}{1-u}\right). \tag{4-7}$$

4.1.2 The median case ($p = 0.5,\ d = 0.5$):

Employing the quantile function for the logistic distribution, then the first four LQ-moments for the logistic distribution will be given as follows:

$$\xi_1 = \alpha + \left[Q_\circ(0.5)\right], \tag{4-8}$$

$$\xi_2 = \frac{1}{2}\left[Q_\circ(0.707) - Q_\circ(0.293)\right], \tag{4-9}$$

$$\xi_3 = \frac{1}{3}\left[Q_\circ(0.794) - 2Q_\circ(0.5) + Q_\circ(0.206)\right], \tag{4-10}$$

and

$$\xi_4 = \frac{1}{4}\left[Q_\circ(0.841) - 3Q_\circ(0.614) + 3Q_\circ(0.386) - Q_\circ(0.159)\right]. \tag{4-11}$$

4.1.3 The Gastwirth case ($p = 0.3,\ d = 1/3$):

Employing the quantile function for the logistic distribution, then the first four LQ-moments for the logistic distribution will be given as follows:

$$\xi_1 = \alpha + \frac{1}{10}\left[3Q_\circ(0.333) + 4Q_\circ(0.5) + 3Q_\circ(0.667)\right], \tag{4-12}$$

$$\xi_2 = \frac{1}{20}\left[3Q_\circ(0.816) + 4Q_\circ(0.707) + 3Q_\circ(0.577) - 3Q_\circ(0.423) - 4Q_\circ(0.293) - 3Q_\circ(0.184)\right], \tag{4-13}$$

$$\xi_3 = \frac{1}{30}\left[3Q_\circ(0.874) + 4Q_\circ(0.794) + 3Q_\circ(0.693) - 6Q_\circ(0.613) - 8Q_\circ(0.5) - 6Q_\circ(0.387) + 3Q_\circ(0.307) + 4Q_\circ(0.206) + 3Q_\circ(0.126)\right], \tag{4-14}$$

and

$$\xi_4 = \frac{1}{40}\left[3Q_\circ(0.904) + 4Q_\circ(0.841) + 3Q_\circ(0.760) - 9Q_\circ(0.709) - 12Q_\circ(0.614) + 9Q_\circ(0.514) - 9Q_\circ(0.486) + 12Q_\circ(0.386) + 9Q_\circ(0.291) - 3Q_\circ(0.240) - 4Q_\circ(0.159) - 3Q_\circ(0.096)\right]. \quad (4\text{-}15)$$

Then, the LQ-kurtosis and the LQ-skewness for each case (trimean, median and Gastwirth) for the logistic distribution can be given by using the conclusions for the first four LQ-moments of the logistic distribution.

4.2 LQ-Moments Estimators

To evaluate the unknown parameters α and β of the logistic distribution employing the LQ-moments, the first two sample LQ-moments for the logistic distribution will be given by employing the following definition of the rth sample LQ-moments:

$$\hat{\zeta}_r = r^{-1}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\hat{\tau}_{p,d}(X_{(r-k:r)}), \qquad r = 1,2,... \quad (4\text{-}16)$$

where

$$\hat{\tau}_{p,d}(X_{(r-k:r)}) = p\hat{Q}_{X_{(r-k:r)}}(d) + (1-2p)\hat{Q}_{X_{(r-k:r)}}(1/2) + p\hat{Q}_{X_{(r-k:r)}}(1-d). \quad (4\text{-}17)$$

$\hat{\tau}_{p,d}(X_{(r-k:r)})$ is the quick estimator of the location of the distribution of $X_{(r-k:r)}$ in a random sample of size r, and $\hat{Q}_X(.)$ denotes the linear interpolation estimator of Q(u) which given by:

$$\hat{Q}_X(u) = (1-\varepsilon)X_{[n'u]:n} + \varepsilon X_{[n'u]+1:n}, \quad (4\text{-}18)$$

where $\varepsilon = n'u - [n'u]$, $n' = n+1$, and $[n'u]$ denote the integral part of $n'u$. Then, the first two sample LQ-moments will be given by:

$$\hat{\zeta}_1 = \hat{\tau}_{p,d}(X_{(1:1)}), \quad (4\text{-}19)$$

and

$$\hat{\zeta}_2 = \frac{1}{2}\left[\hat{\tau}_{p,d}(X_{(2:2)}) - \hat{\tau}_{p,d}(X_{(1:2)})\right]. \quad (4\text{-}20)$$

By equating the first two population LQ-moments with the first two sample LQ-moments of the logistic distribution for each case (trimean, median and Gastwirth), the LQ-moments estimators for the two unknown parameters α and β will be given for each case.

For the trimean case, the LQ-moments estimates (LQMEt) $\hat{\hat{\alpha}}$ and $\hat{\hat{\beta}}$ for the logistic distribution will be estimated. Since, the first sample LQ-moments $\hat{\xi}_1$ is a function of α, β and the second sample LQ-moments $\hat{\xi}_2$ is a function also of α and β, then solving the equations numerically for $\hat{\xi}_1$ and $\hat{\xi}_2$ to find the LQ-moments estimates $\hat{\hat{\alpha}}$ and $\hat{\hat{\beta}}$, then the estimates will be given by using the following two equations:

$$\hat{\hat{\xi}}_1 = \hat{\hat{\alpha}} + \frac{1}{4}\left[\hat{Q}_\circ(0.25) + 2\hat{Q}_\circ(0.5) + \hat{Q}_\circ(0.75)\right], \text{ where } \hat{Q}_\circ(u) = \hat{\hat{\beta}}\ln\left(\frac{u}{1-u}\right)., \quad (4\text{-}21)$$

and

$$\hat{\hat{\xi}}_2 = \frac{1}{8}\left[\hat{Q}_\circ(0.866) + 2\hat{Q}_\circ(0.707) - 2\hat{Q}_\circ(0.293) - \hat{Q}_\circ(0.134)\right], \quad (4\text{-}22)$$

For the median case, the LQ-moments estimates (LQMEm) $\hat{\hat{\alpha}}$ and $\hat{\hat{\beta}}$ will be given by solving the following two equations:

$$\hat{\hat{\xi}}_1 = \hat{\hat{\alpha}} + \left[\hat{Q}_\circ(0.5)\right], \quad (4\text{-}23)$$

and

$$\hat{\hat{\xi}}_2 = \frac{1}{2}\left[\hat{Q}_\circ(0.707) - \hat{Q}_\circ(0.293)\right], \quad (4\text{-}24)$$

and, for the Gastwirth case, the LQ-moments estimates (LQMEg) $\hat{\hat{\alpha}}$ and $\hat{\hat{\beta}}$ will be given using the following two equations:

$$\hat{\hat{\xi}}_1 = \hat{\hat{\alpha}} + \frac{1}{10}\left[3\hat{Q}_\circ(0.333) + 4\hat{Q}_\circ(0.5) + 3\hat{Q}_\circ(0.667)\right], \quad (4\text{-}25)$$

and

$$\hat{\hat{\xi}}_2 = \frac{1}{20}\left[3\hat{Q}_\circ(0.816) + 4\hat{Q}_\circ(0.707) + 3\hat{Q}_\circ(0.577) - 3\hat{Q}_\circ(0.423) - 4\hat{Q}_\circ(0.293) - 3\hat{Q}_\circ(0.184)\right] \quad (4\text{-}26)$$

5. FUZZY LEAST-SQUARES METHOD

The fuzzy least-squares parameters for the logistic distribution can be obtained by using equation (1-1), then we have the following:

$$\frac{1-F(x;\alpha,\beta)}{F(x;\alpha,\beta)} = e^{-\frac{(x-\alpha)}{\beta}}, \qquad -\infty < x < \infty \quad (5\text{-}1)$$

Taking logarithms of both sides we obtained:

$$\ln\left(\frac{1-F(x;\alpha,\beta)}{F(x;\alpha,\beta)}\right) = -\frac{(x-\alpha)}{\beta}, \qquad -\infty < x < \infty \quad (5\text{-}2)$$

and

$$\ln\left(\frac{1-F(x;\alpha,\beta)}{F(x;\alpha,\beta)}\right) = \frac{\alpha}{\beta} - \frac{1}{\beta}(x), \qquad -\infty < x < \infty \quad (5\text{-}3)$$

Let, $x_{(1)} < x_{(2)} < .... < x_{(n)}$ be the order observations in a random sample of size n from $F(x;\alpha,\beta)$. Then the equation (5-3) will be:

$$\ln\left(\frac{1-F(x_{(i)};\alpha,\beta)}{F(x_{(i)};\alpha,\beta)}\right)=\frac{\alpha}{\beta}-\frac{1}{\beta}\left(x_{(i)}\right) \quad , \qquad i=1,\ 2,\ ....,n \tag{5-4}$$

We can use the estimator for the ordinate of the i^{th} empirical point for the logistic plotting technique as follows:

$$y_i=\ln\left(\frac{1-\hat{F}_i}{\hat{F}_i}\right) \tag{5-5}$$

where $\hat{F}_i$ is a point estimator of $F(x_{(i)};\alpha,\beta)$. Several estimators can be employed, for example, the mean rank estimator $\hat{F}_i=i/(n+1)$, the median rank estimator $\hat{F}_i=(i-3/8)/(n+1/4)$, $\hat{F}_i=(i-1/2)/n$ and $\hat{F}_i=(i-0.3)/(n+0.4)$. In this paper, we will employ the mean rank estimator $\hat{F}_i=i/(n+1)$ to find the fuzzy least squares estimator for the logistic distribution. The least square procedure for evaluating the parameters α and β is given below.

Regression analysis may be employed into the model-fitting of observations. The heterogeneous problem is generally difficult to be handled in the regression model. But the heterogeneity of observations is normally presented in practice. Yang and Ko [13] might take into consideration the heterogeneity of observations because of difference clusters of observations. They first cluster the observations and then use the class memberships as weights in the weighted least-squares estimation, it enables there to treat the heterogeneous problem in the regression model fitting. Considering this kind of idea, Yang and Ko [13] advised the use ofthe cluster fuzzy regression analysis which embeds fuzzy clustering into fuzzy regression model fitting at each phase in the iterations. Given a data set $\{(x_j,y_j),j=1,...,n\}$, now, we want to fit a data set to the cluster-wise fuzzy linear regression model:

$$y_j=a_{0i}+a_{1i}x_j,\ i=1,...,c;\ \ j=1,...,n. \tag{5-6}$$

where a_{0i} and a_{1i} are unknown coefficients. Now, let $\mu_{ij}\in[0,1]$ with $\sum_{i=1}^{c}\mu_{ij}=1$ for all $j=1,...,n$. A notation μ_{ij} is employed to represent the membership of the j^{th} data point $\left(x_j,y_j\right)$ belonging to the i^{th} class. After embedding μ_{ij} to the objective function, one has a cluster-wise objective function:

$$J(\mu,\underline{a_0},\underline{a_1})=\sum_{i=1}^{c}\sum_{j=1}^{n}\mu_{ij}^{m}d^2(a_{0i}+a_{1i}x_j,y_j)=\sum_{i=1}^{c}\sum_{j=1}^{n}\mu_{ij}^{m}(y_j-a_{0i}-a_{1i}x_j)^2 \tag{5-7}$$

where $\mu=(\mu_{ij})_{c\times n}$, $\underline{a_0}=(a_{01},a_{02},...,a_{0c})$, $\underline{a_1}=(a_{11},a_{12},...,a_{1c})$ and $m>1$ is an index of fuzziness. Let $L(\mu,\underline{a_0},\underline{a_1},\underline{\lambda})$ be the Lagrangian with

$$L(\mu,\underline{a_0},\underline{a_1},\underline{\lambda})=J(\mu,\underline{a_0},\underline{a_1})+\sum_{j=1}^{n}\lambda_j(\sum_{i=1}^{c}\mu_{ij}-1), \tag{5-8}$$

where $\underline{\lambda}=(\lambda_1,\lambda_2,...,\lambda_n)$. Set the first derivatives of L related to all parameters equal to zero. The following necessary conditions for minimizing $(\mu,\underline{a_0},\underline{a_1})$ of J are given. That is,

$$a_{1i} = \frac{\sum_{j=1}^{n} \mu_{ij}^{m} x_j \sum_{j=1}^{n} \mu_{ij}^{m} y_j - \sum_{j=1}^{n} \mu_{ij}^{m} x_j y_j \sum_{j=1}^{n} \mu_{ij}^{m}}{\left(\sum_{j=1}^{n} \mu_{ij}^{m} x_j \right)^2 - \sum_{j=1}^{n} \mu_{ij}^{m} \sum_{j=1}^{n} \mu_{ij}^{m} x_j^2}, \; i = 1,\dots,c \qquad \text{(5-9)}$$

$$a_{0i} = \frac{\sum_{j=1}^{n} \mu_{ij}^{m} y_j - a_{1i} \sum_{j=1}^{n} \mu_{ij}^{m} x_j}{\sum_{j=1}^{n} \mu_{ij}^{m}}, \; i = 1,\dots,c. \qquad \text{(5-10)}$$

and

$$\mu_{ij} = \left(\sum_{p=1}^{c} \frac{\left(d^2\left(a_{0i} + a_{1i} x_j, y_j \right) \right)^{1/(m-1)}}{\left(d^2\left(a_{0p} + a_{1p} x_j, y_j \right) \right)^{1/(m-1)}} \right)^{-1}, \; i = 1,\dots,c;, \quad j = 1,\dots,n. \qquad \text{(5-11)}$$

Therefore, a cluster-wise FLS algorithm for computing a minimizer of $J(\mu, \underline{a_0}, \underline{a_1})$ has iterations through the necessary conditions (5-9)-(5-11).

A noise cluster is a cluster which has the noise points or outliers. The notion of a noise cluster introduced by Dave [14] is that all of the points have equal prior possibility of belonging to a noise cluster. However, the "good" points increase their chance of being classified into a "good" cluster as the clustering algorithm progresses. It is hoped that all of the noise points (or outliers) can be dumped into a noise cluster during a clustering algorithm in progress. Therefore, Dave [14] introduced a noise prototype as follows: A point *v* is called a noise prototype in the distance $d(x_j, v) = \delta$ between the data point x_j and *v* are all equal to a constant δ , i.e. $d(x_j, v) = \delta$ for j=1,…,n. Now the noise cluster concept is applied to the cluster-wise FLS. Assume that the cluster (*c*+1) is a noise cluster, then the objective function will be:

$$J^0(\mu, \underline{a_0}, \underline{a_1}) = \sum_{i=1}^{c+1} \sum_{j=1}^{n} \mu_{ij}^{m} d_{ij}^2 \qquad \text{with} \qquad \sum_{i=1}^{c+1} \mu_{ij} = 1, \; j = 1,\dots,n, \qquad \text{(5-12)}$$

and

$$d_{ij}^2 = \begin{cases} d^2\left(a_{0i} + a_{1i} x_j, y_j \right) = \left(y_j - a_{0i} - a_{1i} x_j \right)^2, & i = 1,\dots,c; \; j = 1,\dots,n, \\ \delta^2 & , \; i = c+1; \; j = 1,\dots,n, \end{cases} \qquad \text{(5-13)}$$

where

$$\delta^2 = \gamma \left(\frac{\sum_{i=1}^{c} \sum_{j=1}^{n} d_{ij}^2}{nc} \right), \qquad \text{(5-14)}$$

$\gamma > 0$ is a constant. Thus, when c=1, the algorithm with a noise cluster is iterated with the necessary conditions (5-9) and (5-10) and also with

$$\mu_{ij} = \left(\sum_{p=1}^{c+1} \frac{\left(d_{ij}^2\right)^{1/(m-1)}}{\left(d_{pj}^2\right)^{1/(m-1)}} \right)^{-1}, \quad i = 1,...,c+1;, \quad j = 1,...,n. \tag{5-15}$$

Thus, when c=1, the algorithm becomes a robust FLS algorithm for cluster-wise fuzzy regression modal $y_j = a_{01} + a_{11}x_j, \; j = 1,...,n.$ This is because outliers will be dumped to a noise cluster according to the weight of its membership. This algorithm is used to estimate the logistic parameters as follows. In general, we can choose $\gamma = 1$ and the index of fuzziness m = 2. Let

$$y_j = \ln\left(\frac{1 - \hat{F}_j}{\hat{F}_j} \right), \qquad j = 1,...,n \tag{5-16}$$

In the cluster-wise fuzzy regression model $y_j = a_{01} + a_{11}x_j, \; j = 1,...,n.$ then the FLS estimators will be:

$$\hat{\alpha} = -\frac{\hat{a}_{01}}{\hat{a}_{11}}, \quad \text{and} \quad \hat{\beta} = -\frac{1}{\hat{a}_{11}}. \tag{5-17}$$

6. A SIMULATION STUDY OF THE LOGISTIC DISTRIBUTION

A simulation study will be introduced to compare between the properties of fuzzy least square estimators (FLSEs) with different estimators: maximum likelihood estimators (MLEs), TL-moment estimators (TLMEs), L-moment estimators (LMEs) and the three LQ-moment estimators {LQMEt (trimean), LQMEm (median) and LQMEg (Gastwirth)} for the two unknown parameters of the logistic distribution. Comparison will be mainly based on their biases and root mean squared errors (RMSEs). The simulation experiments are performed using the Mathcad (14) software, different sample sizes 10, 30 and 50, and different values for the location parameter $\alpha = -3, -1, 1,$ and 3 and for $\beta = 3$.

For each combination of the sample size and the shape parameters values, the experiment will be repeated 10,000 times. In each experiment, the biases and RMSEs for the estimates of α and β will be obtained and listed in (Tables 1 and 2).

7. RESULTS AND CONCLUSION

It is observed in (Table 1) that the fuzzy least square estimators (FLSEs) are the minimum RMSEs for all different values of α and for all values of n = 30 and 50 are considered here. For n = 10, the MLEs are the minimum RMSEs for all different values of α. As far as biases are concerned, the MLEs are less unbiased for some values of α and n = 10 which are considered here and LMEs for n = 30, and for n = 50, TLMEs and LQMEs are less unbiased. The MLEs and TLMEs be the next one respectively, after the FLSEs which has the minimum RMSEs for n = 50 for estimating α for the logistic distribution. Also, it is observed in (Table 1) that the RMSEs of the TLMEs and the MLEs are also quite close to the FLSEs. Comparing all the methods, we conclude that for the parameter α, the FLSEs should be used for estimating α for the logistic distribution for large sample size.

Table 1. Biases and RMSEs of the parameter estimators for the MLEs, LMEs, TLMEs, LQMEs, and FLSEs for different types of moments for α

n = 10	MLE	LME	TLME	LQMEm	LQMEt	LQMEg	FLSE
$\alpha=-3$	0.00522	0.01055	-0.01399	-0.00378*	-0.01309	-0.00843	-0.00969
	(1.65607)*	(1.73328)	(1.67291)	(1.82456)	(1.68228)	(1.70014)	(1.70548)
$\alpha=-1$	0.02098	0.00188*	-0.02284	-0.01715	-0.02194	-0.0228	-0.03313
	(1.63319)*	(1.74460)	(1.69018)	1.84404	1.69888	1.72387	1.73367
$\alpha=1$	0.02221	0.01221	0.00465	0.02828	0.00372*	0.00557	0.05536
	(1.65914)*	(1.71463)	(1.67793)	(1.80289)	(1.68588)	(1.70986)	(1.74494)
$\alpha=3$	-0.01486	-0.00634	0.00829	0.00704	0.00965	0.01026	-0.00244*
	(1.67102)*	(1.71489)	(1.67165)	(1.80110)	(1.68022)	(1.69923)	(1.71304)
n = 30	**MLE**	**LME**	**TLME**	**LQMEm**	**LQMEt**	**LQMEg**	**FLSE**
$\alpha=-3$	0.00101	0.00242	0.00324	-0.00089	0.00024*	0.00241	0.02642
	(0.95154)	(0.99199)	(0.94961)	(1.06845)	(0.97223)	(0.98029)	(0.93206)
$\alpha=-1$	-0.00453	-0.00166*	-0.00620	0.00505	-0.00654	-0.00718	-0.01027
	(0.94560)	(0.99210)	(0.95012)	(1.08465)	(0.97258)	(0.98525)	(0.92620)
$\alpha=1$	-0.01013	-0.00121*	0.00607	-0.01149	0.00967	0.00405	0.01986
	(0.94926)	(1.00027)	(0.94925)	(1.08192)	(0.97485)	(0.98436)	(0.94266)
$\alpha=3$	-0.00080*	-0.01551	-0.00644	0.00190	-0.00713	-0.00827	-0.00637
	(0.94755)	(1.00660)	(0.96420)	(1.08967)	(0.98925)	(0.99653)	(0.94625)
n = 50	**MLE**	**LME**	**TLME**	**LQMEm**	**LQMEt**	**LQMEg**	**FLSE**
$\alpha=-3$	0.01018	0.00710	0.00464	0.00177*	0.00499	0.00262	0.00269
	(0.72953)	(0.75973)	(0.73275)	(0.84093)	(0.75702)	(0.76756)	(0.72110)
$\alpha=-1$	0.00118	0.01258	0.00053*	0.00794	-0.00123	-0.00066	-0.00452
	(0.73334)	(0.77198)	(0.74044)	(0.83847)	(0.76292)	(0.77429)	(0.73090)
$\alpha=1$	0.00735	0.00204	-0.00111*	0.011050	-0.00298	-0.00261	-0.00151
	(0.74287)	(0.77574)	(0.74430)	(0.83768)	(0.76504)	(0.77558)	(0.74146)
$\alpha=3$	0.00302	0.00374	-0.00518	-0.00091*	-0.00655	-0.00492	-0.00910
	(0.73273)	(0.77059)	(0.73473)	(0.84955)	(0.75716)	(0.76727)	(0.73118)

The root mean squared errors (RMSEs) are reported in brackets in the table.
**: The least biased value or the least root mean squared errors*

Table 2. Biases and RMSEs of the parameter estimators for the MLEs, LMEs, TLMEs, LQMEs, and FLSEs for different types of moments for β

n = 10	MLE	LME	TLME	LQMEm	LQMEt	LQMEg	FLSE
$\alpha=-3$	-0.22019	-0.00297*	0.01633	0.38577	0.56090	0.41566	0.92789
	(0.94404)	(0.82743)	(0.97087)	(1.32627)	(1.16715)	(1.16628)	(1.51949)
$\alpha=-1$	-0.22051	0.01087*	0.01796	0.38173	0.55245	0.41831	0.88472
	(0.93132)	(0.83486)	(0.97386)	(1.33034)	(1.15686)	(1.17056)	(1.47553)
$\alpha=1$	-0.18385	0.00101	-0.00020*	0.38069	0.53937	0.39754	0.90177
	(0.86038)	(0.83451)	(0.97202)	(1.32198)	(1.15372)	(1.16247)	(1.47630)
$\alpha=3$	-0.17572	-0.00470	0.00152*	0.34266	0.54027	0.39893	0.91215
	(0.80445)	(0.82667)	(0.97347)	(1.31292)	(1.15174)	(1.16404)	(1.49171)
n = 30	**MLE**	**LME**	**TLME**	**LQMEm**	**LQMEt**	**LQMEg**	**FLSE**
	-0.05347	0.00491	-0.00313*	0.11715	0.14489	0.13336	0.38011

$\alpha=-3$	(0.45542)	(0.46548)	(0.50151)	(0.75108)	(0.58402)	(0.62030)	(0.46160)
$\alpha=-1$	-0.05418	0.00767*	0.01015	0.10847	0.16034	0.14888	0.37935
	(0.46537)	(0.46567)	(0.51494)	(0.74841)	(0.60095)	(0.63816)	(0.46541)
	-0.04375	-0.00032*	-0.00385	0.12836	0.14426	0.13237	0.38321
$\alpha=1$	(0.45769)	(0.46358)	(0.51069)	(0.75468)	(0.59218)	(0.63032)	(0.45537)
	-0.06399	0.00988	0.00405*	0.11639	0.15429	0.14348	0.38966
$\alpha=3$	(0.46197)	(0.46753)	(0.51396)	(0.75272)	(0.59955)	(0.63534)	(0.46075)
n = 50	**MLE**	**LME**	**TLME**	**LQMEm**	**LQMEt**	**LQMEg**	**FLSE**
	-0.03944	0.00576	0.00403*	0.07110	0.09238	0.08662	0.25837
$\alpha=-3$	(0.35737)	(0.36070)	(0.38725)	(0.57780)	(0.44885)	(0.48041)	(0.34298)
$\alpha=-1$	-0.03670	-0.00187*	-0.00202	0.07054	0.08425	0.07786	0.26045
	(0.35379)	(0.35903)	(0.38850)	(0.57725)	(0.44944)	(0.47843)	(0.35175)
	-0.03062	0.00464	0.00047*	0.07251	0.08947	0.08304	0.25303
$\alpha=1$	(0.36077)	(0.35653)	(0.38539)	(0.57411)	(0.44347)	(0.47626)	(0.34732)
	-0.03432	0.00146*	0.00328	0.07008	0.09335	0.08509	0.26277
$\alpha=3$	(0.35640)	(0.35570)	(0.39200)	(0.57966)	(0.45457)	(0.48416)	(0.35178)

The root mean squared errors (RMSEs) are reported in brackets in the table.
**: The least biased value or the least root mean squared errors*

Now consider the estimation of β. In this case, it is observed in (Table 2) that the FLSEs are the minimum RMSEs for all different values of α and n = 50 and for n = 30, with $\alpha=1,\ 3$. For n = 10, the LMEs are the minimum RMSEs for all different values of α, for n = 30, the MLEs are quite close to the FLSEs. Also, from (Table 2), it is observed that most of the estimators usually overestimate β except the MLEs and LMEs are under estimat β. As far as biases are concerned, for different values of n, the LMEs and TLMEs are less unbiased. Comparing all the methods, we conclude that for the parameter β, the FLSEs should be used for estimating β for large sample size.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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