Physical Science International Journal
8(4): 1-4, 2015, Article no.PSIJ.18487
ISSN: 2348-0130

SCIENCEDOMAIN *international*
www.sciencedomain.org

Approximations in Divisible Groups: Part II

Jeffery Ezearn[1] and William Obeng-Denteh[1*]

[1]Department of Mathematics, Kwame Nkrumah University of Science and Technology (KNUST), Kumasi, Ghana.

Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/PSIJ/2015/18487
Editor(s):
(1) Yang-Hui He, Professor of Mathematics, City Univ. London, UK and Chang-Jiang Chair Professor in Physics and Qian-Ren Scholar, Nan Kai University, China; & Tutor and Quondam-Socius in Mathematics, Merton College, University of Oxford, UK.
(2) Christian Brosseau, Distinguished Professor, Department of Physics, Université de Bretagne Occidentale, France.
Reviewers:
(1) Jose Rosales-Ortega, University Of Costa Rica, Costa Rica.
(2) Anonymous, Duzce University, Turkey.
Complete Peer review History: http://sciencedomain.org/review-history/12222

Original Research Article

Received 24th April 2015
Accepted 6th August 2015
Published 10th November 2015

ABSTRACT

We verify some assertions in the prequel to this paper, in which certain functions which are referred to as proximity functions were introduced in order to study Dirichlet-type approximations in normed divisible groups and similar groups that enjoy a form of divisibility, for instance *p*-divisible groups.

Keywords: Divisible groups; Cauchy sequences; group norms; proximity functions.

1. INTRODUCTION

A divisible group $(G,.)$ is defined as a group such that for every $g \in \{G\}$ and natural number *n* there is an $h \in \{G\}$ such that $g = h^n := h.h^{n-1}$; informally, we say that G has *n*-th roots for all *n*. A foremost example is the group of rational numbers $\mathbb{Q}$ under addition. Similarly, *p-divisible* group is a group with *p*-th roots. Now let ϖ denote a subset of the prime numbers $\{2,3,5,7,\ldots\}$. In the prequel [1,2] to this paper, we studied the ϖ-*divisible* groups, which are groups with *p*-th roots for all $p \in \varpi$. Archetypal examples are the additive subgroups of $\mathbb{Q}$ given by $\mathbb{Q}\{\varpi\} = \{q \in \mathbb{Q}: p|\mathrm{D}(q) \Rightarrow p \in \varpi\}$ where $\mathrm{D}(q)$ is the denominator of *q*. We say a group is uniquely ϖ-divisible if it is a ϖ-divisible group with unique roots. For more introduction to divisible groups, see the references [1,3-7]. We recall the following definitions given in [2]:

Definition 1.1 (Norm on ϖ-Divisible Groups): For a set of primes ϖ, let $(G,\cdot)$ be a ϖ-divisible group with identity element *e* and let $|\cdot|: \mathbb{Q}\{\varpi\} \rightarrow$

**Corresponding author: Email: wobeng-denteh.cos@knust.edu.gh;*

$\mathbb{R}$ be an absolute value function. Then a function $\|\cdot\|: G \to \mathbb{R}$ is a *norm* on *G* if it satisfies:

i. $\|g\| = 0$ only if $g = e$
ii. $\|gh\| \leq \|g\| + \|h\|$
iii. $\|g^r\| = |r|\|g\|, r \in \mathbb{Q}\{\varpi\}$

The absolute value$|\cdot|: \mathbb{Q}\{\varpi\} \to \mathbb{R}$, essentially via Ostrowski's Theorem [8], is the usual one on the real numbers or on the *p*-adic numbers. We denote by $(G,\cdot,\|\cdot\|)$ a ϖ-divisible group with a norm $\|\cdot\|$.

Definition 1.2 (Proximity Function on Groups): Let *G* be a group with identity *e*. Then a function $\varrho: G\backslash\{e\} \to \mathbb{R}$ is a *proximity function* on *G* if for all $g \neq h$:

i. $\varrho(g \neq e) = \varrho(g^{-1}) > 0$
ii. $\varrho(gh^{-1}) \leq C\varrho(g)\varrho(h)$
iii. $\varrho(gh^{-1}) \leq C\varrho(g)$ if $\varrho(g) = \varrho(h)$

where $C > 0$ is an absolute constant. If in (ii) we have the stronger bound $\varrho(gh^{-1}) \leq C\max\{\varrho(g), \varrho(h)\}$, then we say ϱ is an ultra-metric proximity function. Furthermore, if ϱ is integer-valued with $C = 1$ and that (ii) and (iii) read $\varrho(gh^{-1}) |\operatorname{lcm}(\varrho(g), \varrho(h))$ and $\varrho(gh^{-1})|\varrho(g)$ if $\varrho(g) = \varrho(h)$ respectively, then we say ϱ is an order function.

For Abelian torsion groups G, the function $\varrho(.)$ = ord(.) is an order function (see Example 1.4 in [1] for more examples).

Definition 1.3 (Proximity Function on Normed ϖ-Divisible Groups): Let $(G,\cdot,\|\cdot\|)$ be a normed ϖ-divisible group with identity e and let ϱ be a proximity function on G. Then ϱ is said to be a close proximity function on G if there exists a $\mu_0 > 0$ such that $\inf\{\varrho(g_n)^{\mu}\|g_n\|\} = 0$ for some null sequence $\{g_n\}_{n=1}^{\infty} \subset G\backslash\{e\}$ if and only if $\mu < \mu_0$; otherwise, then ϱ is an open proximity function on G. We shall say that the elements in G are in close proximity (andin close order) to each other; else, where necessary, we shall say the elements are in open proximity (resp. in open order) to each other.

We typify a close proximity function on *G* by $(\varrho; C, \mu_0)$. The main result proved in [2] is the following theorem.

Theorem 1.4: Let $(\varrho; C, \mu_0)$ be a close proximity function on $(G,\cdot,\|\cdot\|)$ and let $g \in G$. Then for every $\mu > \mu_0$ and Cauchy sequence $\{g_n\}_{n=1}^{\infty} \subset G\backslash\{g, e\}$ converging to g, there exists N such that $\|gg_n^{-1}\| = O(\varrho(g_n)^{-\mu})$ if and only if $n \leq N$, where the implied constant is independent of n or g; moreover, this is also true for $\mu = \mu_0$ if $\varrho$is ultra-metric and the implied constant is less than $\frac{1}{C^{\mu_0}}\inf_{g\neq g_n}\{\varrho(gg_n^{-1})^{\mu_0}\|gg_n^{-1}\|\}$.

Theorem 1.4 implies that there can be only finitely many elements of G in close proximity to any element in G with respect to the given estimates; or equivalently, Cauchy sequences in G do not converge inside G with respect to the given estimates. A converse to this theorem, would give a Dirichlet-type approximation for (incomplete) ϖ-divisible groups. In the present paper, we give a sketchy verification of some assertions on examples of proximity functions given in [2]. On the other hand, we have been unable to prove exactly the Dirichlet-type approximation theorem for ϖ-divisible groups and we leave the task to other author(s).

2. PRELIMINARIES

We require the following definitions and results. A norm $\|.\|$ on an arbitrary group G with identity *e* is said to be *discrete* if

(1) $\|.\|: G \to \mathbb{R}_{\geq 0}$
(2) $\|ab\| \leq \|a\| + \|b\|, \forall a, b \in G$
(3) $\|a^n\| = |n|\|a\|, a \in G, n \in \mathbb{Z}$
(4) $\inf_{a\in G\{e\}}\|a\| > 0$

Let $\mathbb{K}$ be an algebraic number field and let $\overline{\mathbb{Q}}$ be the field of algebraic numbers. The absolute Weil height $h: \mathbb{K} \to \mathbb{R}_{\geq 0}$is given by

$$h(\cdot) := \prod_{v} \max\{1, |\cdot|_v\}$$

where v runs through all places of $\mathbb{K}$ and $|\cdot|_v$is a normalised absolute value, hence $\prod_v|\alpha|_v = 1$. We know (see [9]) that $h(\alpha\beta) \leq 2h(\alpha)h(\beta)$ and also $h(\alpha^{-1}) = h(\alpha)$ if $\alpha \neq 0$.

The p-adic norm $|\cdot|_p$of a rational number $q = \frac{a}{b}$, where a, b are integers with $b \neq 0$ is given by

$$|q|_p = p^{-\left(v_p(a) - v_p(b)\right)}$$

Where $p^{v_p(a)}$ is the greatest power dividing a and similarly $p^{v_p(b)}$ is the greatest power dividing b.

3. MAIN RESULTS

We now establish the main result of this paper, which was stated without proof in [2]. The proof here is a sketch.

Lemma 3.1: The following are close proximity functions on the respective groups defined:

(i) Suppose the absolute value function associated to the normed ϖ-divisible group $(G,\cdot,\|\cdot\|)$ is the usual one on the real numbers. Assume S is a normal subgroup of G such that the quotient group G/S is Abelian and torsion, and that the norm$\|\cdot\|$ is a discrete norm on S—i.e., there is an absolute constant l suchthat $\|g \in S\backslash\{e\}\| \geq l$. Then the function $\varrho_{G/S}(g) = \mathrm{ord}(g \cdot S) \coloneqq \min\{n \in \mathbb{Z}_{>0}: g^n \in S\}$ is a close order function on G with $\mu_0 = 1$, $C = 1$; moreover, if ϖ is a singleton set then ϱ is ultra-metric. (We refer to this as a ϖ-ary order function on G).

(ii) Given a prime p and the group $\mathbb{Q}\{p\}$, then the function $\varrho_p(q \neq 0) = \lceil p^{\lfloor \log(|q|_\infty)/\log p \rfloor} \rceil$ (where $\lfloor\cdot\rfloor$ (resp. $\lceil\cdot\rceil$) denotes the floor (resp. ceiling) function and where $|\cdot|_\infty$ is the usual absolute value on the real numbers) is a close ultra-metric proximity function on $\mathbb{Q}\{p\}$ with $\mu_0 = 1$ and $C = p$ given the usual p-adic norm on $\mathbb{Q}$. (We refer to this proximity function as the p-adic proximity function on $\mathbb{Q}\{p\}$).

(iii) For an algebraic number field $\mathbb{K}$ with the usual normalised absolute values $|\cdot|_v$ over all places v such that $\prod_v |\alpha|_v = 1$ for every $\alpha \in \mathbb{K}\backslash\{0\}$, the function $\varrho_{\mathbb{K}}(\alpha) \coloneqq \prod_v \max\{1, |\alpha|_v\}$—i.e., the Weil height—is a close proximity function on $\mathbb{K}^+$ with $\mu_0 = 1$ and $C = 2$ given the norm defined by the usual absolute value on the complex numbers. (We shall refer to this as the $\mathbb{K}$-proximity function).

Proof. The proof of the above lemma would be generally sketchy.

For (i), it is easy to see that since $\varrho_{G/S}(g) = ord(g \cdot S) \coloneqq min\{n \in \mathbb{Z}_{>0}: g^n \in S\}$, that is since $\varrho_{G/S}$ denotes the order of agroup, then straightforwardly, it suffices for the definition of a proximity(indeed, an order function). To see that it is a close order function, we let $\{g_n\}_{n\geq 1} \subset G\backslash\{e\}$be any null sequence; then we observe that for$\mu \geq \mu_0$ = 1, we have

$$\inf\{\varrho(g_n)^\mu \|g_n\|\} \geq \inf\|g_n\| > 0$$

which is so since $\varrho(g_n) \geq 1$.

For (ii), we observe that for $q \neq r$ and $q, r \neq 0$, we have

$$\varrho_p(q) = \lceil p^{\lfloor log(|q|_\infty)/\, log\, p \rfloor} \rceil = \lceil p^{\lfloor log(|-q|_\infty)/\, log\, p \rfloor} \rceil = \varrho_p(-q)$$

and

$$\begin{aligned} \varrho_p(q-r) &= \lceil p^{\lfloor log(|q-r|_\infty)/\, log\, p \rfloor} \rceil \\ &\leq \lceil p^{\lfloor log(|q|_\infty)+log(|r|_\infty)/\, log\, p \rfloor} \rceil \\ &\leq \lceil p^{1+\lfloor log(|q|_\infty)+log(|r|_\infty)/\, log\, p \rfloor} \rceil \\ &\leq p\lceil p^{\lfloor log(|q|_\infty)/\, log\, p \rfloor} \rceil \lceil p^{\lfloor log(|r|_\infty)/\, log\, p \rfloor} \rceil \\ &= p\varrho_p(q)\varrho_p(r) \end{aligned}$$

If $\varrho_p(q) = \varrho_p(r)$, we easily see that $\varrho_p(q-r) \leq p\varrho_p(q)$. Finally, if$\{q_n\}_{n\geq 1} \subset \mathbb{Q}\{p\}$ is a non-zero null sequence, the we see that for all$\mu \geq \mu_0$= 1 and with the p-adic norm $|.|_p$, we have

$$\inf\{\varrho_p(q_n)^\mu |q_n|_p\} \geq 1$$

which is so since by definition we have the inequality $\varrho_p(q) \geq |q|_p^{-1}$.

For (iii), we know that

$$\varrho_{\mathbb{K}}(\alpha) = \varrho_{\mathbb{K}}(\alpha^{-1})$$

and that

$$\varrho_{\mathbb{K}}(\alpha\beta^{-1}) \leq 2\varrho_{\mathbb{K}}(\alpha)\varrho_{\mathbb{K}}(\beta^{-1}) = 2\varrho_{\mathbb{K}}(\alpha)\varrho_{\mathbb{K}}(\beta)$$

It is easy to see that $\varrho_{\mathbb{K}}(\alpha\beta^{-1}) \leq 2\varrho_{\mathbb{K}}(\alpha)$ when $\varrho_{\mathbb{K}}(\alpha) = \varrho_{\mathbb{K}}(\beta)$. Finally, if $\{\alpha_n\}_{n\geq 1} \subset \mathbb{K}$ is a non-zero null sequence, then for all $\mu \geq \mu_0$ = 1 and norm $|.|$, we have

$$\inf\{\varrho_{\mathbb{K}}(\alpha_n)^\mu |\alpha_n|\} \geq 1$$

which is so since normalisation of absolute values implies that

$$|\alpha_n|\varrho_{\mathbb{K}}(\alpha_n) \prod_{\substack{v \\ |\alpha_n|_v < 1}} |\alpha_n|_v = 1$$

which completes the proof.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

REFERENCES

1. Baumslag G. Some aspects of groups with unique roots. PhD Thesis, University of Manchester; 1958.
2. Ezearn J, Obeng-Denteh W. Approximations in divisible groups: Part I, Physical Sciences Journal International. 2015;6 (2):112-118.
3. Feigelstock, Shalom. Divisible is injective. Soochow J. Math. 2006;32(2):241–243
4. Griffith, Phillip A. Infinite Abelian group theory. Chicago Lectures in Mathematics. University of Chicago Press; 1970.
5. Hurwitz A. Ueber die angenäherte Darstellung der Irrationalzahlen durch rationale Brüche. Mathematische Annalen. 1891;39(2):279–284.
6. Lang S. Algebra, Second Edition. Menlo Park, California: Addison-Wesley; 1984.
7. Mal'cev AI. On a class of homogenous spaces. Izvestiya Akad. Nauk SSSR. Ser. Mat. 1949;**13**:201-212.
8. Ostrowski A. Über einige Lösungen der Funktionalgleichung $\varphi(x)\cdot\varphi(y)=\varphi(xy)$. Acta Mathematica (2nd ed.). 1916;41(1):271–284.
9. Waldschmidt M. Diophantine approximation on linear algebraic groups. Grundlehren 326, Springer; 2000.

Peer-review history:
The peer review history for this paper can be accessed here:
http://sciencedomain.org/review-history/12222