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Fixed Point Theorems in Quasi-2-Banach Spaces

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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ABSTRACT

A number of authors have studied various aspects of fixed point theory in the setting of 2-metric and 2-Banach spaces. In this paper we prove a fixed point theorem for mappings in quasi-2-Banach space via an implicit relation. The results of this paper extend a host of previously known results for metric space in a quasi-2-Banach space.

Keywords: Cauchy sequence; quasi-2-banach space; fixed point.

1. INTRODUCTION

Gahler [1] initiated the concepts of 2-metric and 2-Banach space and Iseki in [2,3], obtained basic results on fixed points in such spaces. These new spaces have subsequently been studied by several mathematicians (for example [4,5,6,7,8]). Recently [8], also proved some results in 2-Banach spaces. In 2006, Park [9] introduces the concepts of quasi-2-normed space and quasi-(2; p)-normed space. In this paper we prove a fixed point theorem for mappings in quasi-2-Banach space via an implicit relation.

We start with some definitions before presenting main theorem.

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Definition 1.1 [1] Let *X* be a real linear space of dimension greater than 1 and let $\|\cdot,\cdot\|$ be a real valued function on $X \times X$ satisfying the following four conditions:

(2 N_1) $\|x, y\| = 0$ if and only if *x* and *y* are linearly dependent in *X*,
(2 N_2) $\|x, y\| = \|y, x\|$ for all *x*, $y \in X$,
(2 N_3) $\|x, \alpha y\| = |\alpha| \cdot \|x, y\|$ for every real number α;
(2 N_4) $\|x, y+z\| \le \|x, y\| + \|x, z\|$ for all *x*, *y*, $z \in X$.

The function $\|\cdot,\cdot\|$ is called a 2-norm on *X* and the pair $(X, \|\cdot,\cdot\|)$ is called a linear 2-normed space. So a 2- norm $\|x, y\|$ always satisfies $\|x, y+\alpha x\| = \|x, y\|$, for all *x*, $y \in X$ and all scalars α. We cite some examples of 2-Banach spaces from the current literature (see [10], [11]).

Example 1.2 Let $X = R^3$ and consider the following 2-norm on *X* as

$$\|x, y\| = \left| \det \begin{Vmatrix} i & j & k \\ a & b & c \\ d & e & f \end{Vmatrix} \right| = \left[(bf-ce)^2 + (cd-af)^2 + (ae-db)^2 \right]^{1/2},$$

where $x = ai + bj + ck$ and $y = di + ej + fk$. Then $(X, \|\cdot,\cdot\|)$ is a 2-Banach space.

Example 1.3 Let *X* is Q^3, where Q is the field of rational number and consider the following 2-norm on *X* as:

$$\| x, y \| = \left| \det \begin{Vmatrix} i & j & k \\ a & b & c \\ d & e & f \end{Vmatrix} \right|,$$

where $x = ai + bj + ck$ and $y = di + ej + fk$. Then $(X, \|\cdot,\cdot\|)$ is not a 2-Banach space.

Definition 1.4 [9] Let *X* be a linear space. A *quasi-2-normed* is a real valued function on $X \times X$ satisfying three conditions of Definition 2: (2 N_1), (2 N_2), (2 N_3) and the condition (2 $N_4^{\bullet}$): There is a constant $k \ge 1$ such that $\|x+y, z\| \le k\|x, z\| + k\|y, z\|$ for all $x, y, z \in X$.

The pair $(X, \|\cdot,\cdot\|)$ is called a *quasi-2-normed space* if $\|\cdot,\cdot\|$ is a quasi-2-norm on *X.* The smallest possible *k* is called the modulus of concavity of $\|\cdot,\cdot\|$.

A quasi-2-norm $\|\cdot,\cdot\|$ is called a *quasi-*(2; *p*)*-norm* ($0 < p \le 1$) if $\|x+y, z\|^p \le \|x, z\|^p + \|y, z\|^p$ for all $x, y, z \in X$.

Definition 1.5 A sequence $\{x_n\}$ in a quasi-2-norm space $(X,\|\cdot,\cdot\|)$ is said to be a *Cauchy sequence* if $\lim_{m,n\to\infty}\|x_m - x_n, u\| = 0$ for all *u* in *X*. (Symbolically we denote $d(x_m, x_n) = \|x_m - x_n, u\|$)

Definition 1.6 A sequence $\{x_n\}$ in a quasi-2-norm space $(X,\|\cdot,\cdot\|)$ is said to be *convergent* if there is a point *x* in *X* such that $\lim_{n\to\infty}\|x_n - x, y\| = 0$ for all *y* in *X*. If $\{x_n\}$ converges to *X*, we write $\{x_n\} \to x \ as\, n \to \infty$.

Definition 1.7 A linear quasi-2-norm space $(X,\|\cdot,\cdot\|)$ is said to be *complete* if every Cauchy sequence is convergent to an element of *X*.

Definition 1.8 A complete quasi-2-norm space is called a *quasi-2-Banach space*.

Definition 1.9 Let *X* be a quasi-2-Banach space and *T* be a self-mapping of *X*. *T* is said to be *continuous at X* if for every sequence $\{x_n\}$ in *X*, $\{x_n\} \to x$ as $n \to \infty$ implies $\{T(x_n)\} \to T(x)$ as $n \to \infty$.

We also need the following notion from [12].

Definition 1.10 The set of all upper semi-continuous functions with 5 variables $f : R_+^5 \to R$ satisfying the properties:

(a). f is non decreasing with respect to each variable,
(b). $f(t,t,t,t,t) \le t,\ t \in R_+$,

will be noted F_5 and every such function will be called a F_5-function.

Some examples of F_5-function are as follows:

1. $f(t_1,t_2,t_3,t_4,t_5) = \max\{t_1,t_2,t_3,t_4,t_5\}$,
2. $f(t_1,t_2,t_3,t_4,t_5) = [\max\{t_1t_2, t_2t_3, t_3t_4, t_4t_5, t_5t_1\}]^{1/2}$,
3. $f(t_1,t_2,t_3,t_4,t_5) = [\max\{t_1^p, t_2^p, t_3^p, t_4^p, t_5^p\}]^{1/p}$, $p > 0$,
4. $f(t_1,t_2,t_3,t_4,t_5) = (a_1t_1^p + a_2t_2^p + a_3t_3^p + a_4t_4^p + a_5t_5^p)^{1/p}$,
   where $p > 0$ and $0 \le a_i, \sum_{i=1}^{5} a_i \le 1$,
5. $f(t_1,t_2,t_3,t_4,t_5) = \dfrac{t_1+t_2+t_3}{3}$ or $f(t_1,t_2,t_3,t_4,t_5) = \dfrac{t_1+t_2}{2}$ etc.

2. MATERIALS AND METHODS

We state the following lemma which we will use for the proof of the main theorem.

Lemma 2.1 Let $(X,\|\cdot,\cdot\|)$ be a quasi-2-normed space with the coefficients $k \geq 1$ and $\{x_n\}$ is a sequence in *X*. If $d(x_n, x_{n+1}) \leq c^n l,\ 0 \leq c < \frac{1}{k} \leq 1,\ l \geq 0,\ \ \forall n \in N$, then $\{x_n\}$ is a Cauchy sequence.

Proof:

$$\begin{aligned}\|x_n - x_{n+m}, u\| &\leq k\|x_n - x_{n+1}, u\| + k\|x_{n+1} - x_{n+m}, u\|)\\ &\leq k\|x_n - x_{n+1}, u\| + k^2\|x_{n+1} - x_{n+2}, u\| + k^2\|x_{n+2} - x_{n+m}, u\| \leq \ldots\\ &\leq k\|x_n - x_{n+1}, u\| + k^2\|x_{n+1} - x_{n+2}, u\| + k^3\|x_{n+2} - x_{n+3}, u\| + \ldots\\ &\quad + k^{m-2}\|x_{n+m-3} - x_{n+m-2}, u\| + k^{m-1}\|x_{n+m-2} - x_{n+m-1}, u\| +\\ &\quad + k^{m-1}\|x_{n+m-1} - x_{n+m}, u\|\\ &\leq kc^n l + k^2 c^{n+1} l + k^3 c^{n+2} l + \ldots + k^{m-1} c^{n+m-2} l + k^m c^{n+m-1} l\\ &\leq kc^n l \tfrac{1-(kc)^m}{1-kc} \leq kc^n l \tfrac{1-(kc)^m}{1-kc} < \tfrac{kc^n l}{1-kc}.\end{aligned}$$

And so $\lim\limits_{n\to\infty}\|x_n - x_{n+m}, u\| = 0$. It implies that $\{x_n\}$ is a Cauchy sequence in *X*. This completes the proof of the lemma.

Theorem 2.2 Let *X* be a quasi-2-Banach space with the coefficients $k \geq 1$ and $f \in \mathrm{F}_5$. Let $T : X \to X$ satisfying

$$\|T(x) - T(y), u\| \leq cf(\|x - y, u\|, \|x - Tx, u\|, \|y - Ty, u\|, \left\|y - T^2 x, u\right\|, \|y - Tx, u\|), \tag{1}$$

for each $x, y, u \in X$ and $0 \leq c < \frac{1}{k} \leq 1$. Then *T* has a unique fixed point *z* in *X* such that $x_0 \in X$ gives $\lim\limits_{n\to\infty} T^n(x_0) = z$.

Proof. Let x_0 be an arbitrary point in *X*. Define the sequences $\{x_n\}$ as follows:

$$x_n = Tx_{n-1} = T^n x_0,\ n = 1, 2, \ldots.$$

Take $u \in X$. Denote

$$d_n(u) = \|x_n - x_{n+1}, u\|,\ n = 0, 1, 2, \ldots.$$

By the inequality (1) we get:

$$\begin{aligned} d_n(u) &= \|x_n - x_{n+1}, u\| = \|T^n x_0 - T^{n+1} x_0, u\| \\ &\le cf(\|T^{n-1}x_0 - T^n x_0, u\|, \|T^{n-1}x_0 - T^n x_0, u\|, \|T^n x_0 - T^{n+1}x_0, u\|, \\ &\|T^n x_0 - T^{n+1} x_0, u\|, \|T^n x_0 - T^n x_0, u\|) \\ &= cf[d_{n-1}(u), d_{n-1}(u), d_n(u), d_n(u), 0]. \end{aligned}$$

By this inequality and the properties of *f*, it follows

$$d_n(u) \le c d_{n-1}(u)$$

In general, we have

$$d_n(u) \le c^n d_0(u) = c^n l, n \in N, \tag{2}$$

where $l = d_0(u) = \|x_0 - x_1, u\|$, and so

$$\lim_{n\to\infty} d_n(u) = \lim_{n\to\infty} \|x_n - x_{n+1}, u\| = 0 \,. \tag{3}$$

Then, from (2) and Lemma 2.1 is derived that $\{x_n\}$ is a Cauchy sequence in *X* and hence is convergent in *X*. Let $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} T^n x_n = \alpha \in X$. The limit α is unique. Assume that $\alpha' \ne \alpha$ and $\alpha' = \lim\limits_{n\to\infty} x_n$. Then by condition (2 $N_4^\bullet$) of Definition 1.5, we obtain

$$\|\alpha - \alpha', u\| \le k\|\alpha - x_n, u\| + k\|x_n - \alpha', u\|.$$

Letting *n* tend to infinity we get $\|\alpha - \alpha', u\| = 0$ for all $u \in X$ and so $\alpha = \alpha'$.

Let us prove now that α is a fixed point of T. Assume that $\alpha \ne T\alpha$. Then, by Definition 1.3, we obtain $\|\alpha - T\alpha, u\| \le k\|\alpha - x_n, u\| + k\|x_n - T\alpha, u\|$. Then, if $n \to \infty$, we get

$$\|\alpha - T\alpha, u\| \le k \overline{\lim_{n\to\infty}} \|x_n - T\alpha, u\| \,. \tag{4}$$

From (1), we get

$$\begin{aligned} &\|x_n - T\alpha, u\| = \|Tx_{n-1} - T\alpha, u\| \\ &\le cf(\|x_{n-1} - \alpha, u\|, \|x_{n-1} - Tx_{n-1}, u\|, \|\alpha - T\alpha, u\|, \|\alpha - T^2 x_{n-1}, u\|, \|\alpha - Tx_{n-1}, u\|) \\ &= cf(\|x_{n-1} - \alpha, u\|, \|x_{n-1} - x_n, u\|, \|\alpha - T\alpha, u\|, \|\alpha - x_{n+1}, u\|, \|\alpha - x_n, u\|) \end{aligned}$$

.

Letting *n* tend to infinity we have

$$\overline{\lim_{n\to\infty}}\|x_n - T\alpha, u\| \le cf(0,0,\|\alpha - T\alpha, u\|, 0, 0) \le c\|\alpha - T\alpha, u\|. \tag{5}$$

From (4) and (5), we have

$$\|\alpha - T\alpha, u\| \le k\,\overline{\lim_{n\to\infty}}\|x_n - T\alpha, u\| \le k\,c\|\alpha - T\alpha, u\|.$$

Since $0 < c < \frac{1}{k} < 1$ we have $\|\alpha - T\alpha, u\| = 0$ for all $u \in X$. So α is a fixed point of *T.*

Let we prove now the uniqueness. Assume that $\alpha' \neq \alpha$ is also a fixed point of T .

By (1) for $x = \alpha$ and $y = \alpha'$ we get:

$$\begin{aligned}
&\|\alpha - \alpha', u\| = \|T(\alpha) - T(\alpha'), u\| \\
&\le cf(\|\alpha - \alpha', u\|, \|\alpha - T\alpha, u\|, \|\alpha' - T\alpha', u\|, \|\alpha' - T^2\alpha, u\|, \|\alpha' - T\alpha, u\|) \cdot \\
&= cf(\|\alpha - \alpha', u\|, 0, 0, \|\alpha' - \alpha, u\|, \|\alpha' - \alpha, u\|) \le c\|\alpha - \alpha', u\|.
\end{aligned}$$

And so, we have

$$\|\alpha - \alpha', u\| \le c\|\alpha - \alpha', u\|. \tag{6}$$

By (6) we get: $\|\alpha - \alpha', u\|$=0. Thus, we have again $\alpha = \alpha'$. This completes the proof of the theorem.

3. RESULTS AND DISCUSSION

For different expressions of *f* in Theorem 2.2 we get different theorems. In case $f(t_1, t_2, t_3, t_4, t_5) = t_1$ we have an extension of Banach's contraction principle for metric space in a quasi-2-Banach space:

Corollary 3.1 Let $(X, \|\cdot,\cdot\|)$ be a quasi-2-Banach space with coefficient $k \ge 1$ and $T : X \to X$ be a mapping such that

$$\|Tx - Ty, u\| \le c\|x - y, u\|$$

for all $x, y \in X$, where $0 \le c < \dfrac{1}{k}$. Then *T* has a unique fixed point α in *X* such that $x_0 \in X$ gives $\lim_{n\to\infty} T^n(x_0) = \alpha$ **.**

In case $f(t_1,t_2,t_3,t_4,t_5)=\frac{t_2+t_3}{2}$ we have an extension of Kannan's contraction principle for metric space [13] in a quasi-2-Banach space:

Corollary 3.2 Let $(X,\|\cdot,\cdot\|)$ be a quasi-2-Banach space with coefficient $k\geq 1$ and $T:X\to X$ be a mapping such that

$$\|Tx-Ty,u\|\leq c(\|x-Tx,u\|+\|y-Ty,u\|)$$

for all $x,y\in X$, where $0\leq c<\frac{1}{2k}$. Then *T* has a unique fixed point α in *X* such that $x_0\in X$ gives $\lim_{n\to\infty}T^n(x_0)=\alpha$.

Corollary 3.3 For $f(t_1,t_2,t_3,t_4,t_5)=\max\{t_2,t_3\}$ we have an extension of Bianchini's contraction principle for metric space [14] in a quasi-2-Banach space.

Corollary 3.4 For $f(t_1,t_2,t_3,t_4,t_5)=\frac{at_1+bt_2+ct_3}{a+b+c}$ where *a*, *b*, *c* are nonnegative numbers such that $a+b+c<1$, we have an extension of Reich's contraction principle for metric space [9] in a quasi-2-Banach space.

Remark 1: For different *f*, we can obtain many other similar results of Rhoades classification [15,16].

Remark 2: For $k=1$ we take our main theorem and its corollaries for 2-Banach spaces.

4. CONCLUSION

In this paper we proved fixed point theorems for mappings in quasi-2-Banach space via an implicit relation. The results of this paper extend the previously known results for metric space in a quasi-2-Banach space.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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