



## New Aspects of Double Bounded Polynomial Homotopy

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## Abstract

This work presents some new aspects about the mathematical properties of the double bounded polynomial homotopy like: isolated symmetry branches equation and a study about initial and final points of the path and curvature radius. The application of the homotopy is illustrated by solving a mathematical problem and a nonlinear circuit.

*Keywords:* Homotopy continuation methods; multiple operating points

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## 1 Introduction

The task of finding Direct Current (DC) solutions is important because this analysis serves as the starting point for other well-known kinds of analyses such as the small-signal analysis. In plain words, DC analysis consists of finding the solutions of a system of nonlinear algebraic equations (NAEs) that constitute the equilibrium equation of a nonlinear static circuit [10, 5, 8].

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The rapid increase on the number of transistors in an integrated circuit and the increase of complexity for the models (result of lowering the dimension in the components) results in highly complex NAEs leading to two phenomena: the existence of several unexpected operating points and the lack of convergence to the operating point (OP) by the Newton-Raphson (NR) method (employed in circuit simulators). The existence of multiple operating points generates the need to use other methods, other than the NR, like homotopy methods [1, 2, 12, 6, 18, 17]. Opposite to the NR method, homotopy is capable to locate multiple operating points. Besides, the high complexity of the equilibrium equation makes NR, sometimes, unable to converge mainly because this method is locally convergent.

The homotopy method reported by [9] is a highly efficient multi-parameter method [19] that locates the OP for large circuits containing only MOS transistors. In [7], an excellent revision of globally convergent probability-one homotopy methods applied to circuit simulation with bipolar transistors is presented. All the homotopy methods [9, 19, 7] described above have been proved useful to locate one or more operating points that converge to solutions where the NR method is unable to calculate. Nevertheless, HCM methods have some disadvantages; one of them is that, in general, they do not have a reliable stop criterion [16]. It means that it is not possible to determine, with mathematical certainty, when to stop looking for new solutions on the homotopy path. This fact leads to the waste of numerical continuation iterations looking for solutions that do not exist. On this regard, in [3], a technique is proposed to restrict the search space for certain group of circuits including bipolar transistors, ensuring a stop criterion. Nevertheless, that type of homotopy requires advanced knowledge about multistable circuits to use it effectively. Moreover, double bounded homotopy (DBH) [14] and double bounded polynomial homotopy (DBPH) [16] possess a formal stop criterion and can be applied to multistable circuits containing diodes, tunnel diodes, BJT transistors, MOS transistors, among others. The main property of these homotopies is that the path is closed around a symmetry axis. This characteristic can be used to create a formal stop criterion by the path following [15] of the half of the trajectory. Therefore, this work presents some new aspects about the mathematical properties of the DBPH homotopy [16] like: isolated symmetry branches equation and a study about initial and final points of the path and curvature radius. The application of the homotopy is illustrated by solving a mathematical problem and a nonlinear circuit.

This paper is organized as follows. In Section 2, we briefly explain the basic concepts of the homotopy methods related to the proposal for this work. In Section 3, we present the basic idea of the double bounded homotopy concept. In Section 4, we explain in detail the DBPH homotopy including some new aspects. In Section 5, we present some study cases. In Section 6, we summarize our findings and suggest possible directions for future investigations. Finally, a brief conclusion is given in Section 7.

## 2 Basic idea of HCM methods

It is evident that alternate methods must be used in order to solve problems originated by applying the NR method. Among them, HCM methods can cope with the problem of multiple DC operating points. Homotopy methods [9, 13, 4, 19, 7] have been proved useful to locate multiple operating points and converge to solutions where the NR method is incapable. Nevertheless, homotopy methods are affected by certain inconveniences, among them are: a formal stop criterion and computing time.

Solving the nonlinear algebraic system

$$\mathbf{f}(\mathbf{x}) = \mathbf{0}, \quad (2.1)$$

resorting to a plain homotopy formulation implies obtaining an augmented version of the equilibrium equation in the form

$$\mathbf{H}(\mathbf{f}(\mathbf{x}), \lambda) = \mathbf{0}, \quad (2.2)$$

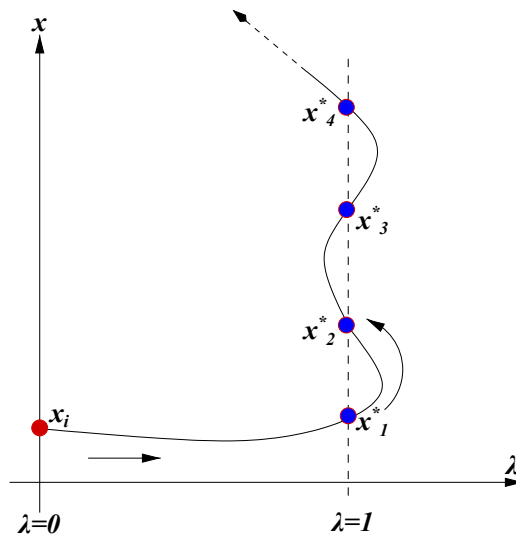


Figure 1: Path for Newton's homotopy.

where  $\lambda$  is the homotopy parameter and  $H^{-1}(0)$  the family of solutions that conforms the homotopy path.

An example of homotopy formulation is Newton's homotopy

$$H(f(x), \lambda) = f(x) + (\lambda - 1)f(x_i) = 0, \tag{2.3}$$

where  $x_i$  is the starting point of the trajectory.

This system has the following properties:

1. At initial value  $\lambda = 0$ ,

$$H(f(x), 0) = H(f(x), 0) = f(x) - f(x_i) = 0, \tag{2.4}$$

where the system admits at least the solution  $x_i$ .

2. At value  $\lambda = 1$ ,

$$H(f(x), 1) = H(f(x), 1) = f(x) = 0, \tag{2.5}$$

the system has the solutions of the original system from equation (2.1).

Fig. 1 shows the homotopy path for the scheme above. Line  $\lambda = 1$  is called *the solution line* of the homotopy and every time the line is crossed by the homotopy path, one or more solutions are found ( $x_1^*, x_2^*, x_3^*, x_4^*, \dots$ ). In particular, this scheme exhibits serious drawbacks regarding the lack of a stop criterion, prone to numerical problems due to anomalous behaviour of the paths, such as, folding, bifurcations, abbreviation, and disconnections on the paths [19].

Nevertheless, the main issue regarding homotopy schemes, such as Newton's homotopy, consists on the lack of a reliable stop criteria which means that thumb rules must be applied. Homotopy paths may appear in two ways: open paths and closed paths. For the first case, establishing a stop criterion is very difficult to do, one usually resorts to set a maximum number of homotopy iterations. As for the second case, the stop criterion makes use of the closure of the path in order to assess whether or not the current iteration crosses an already found point.

### 3 The basic idea of the Double Bounded Homotopy concept

In [14, 16] was introduced the idea to create a double bounded homotopy; a way to cope with the stop criterion is to bend the open solution path in order to convert the open path into a closed path as depicted in Fig. 2. This is achieved by setting two solution lines at  $\lambda = a$  and  $\lambda = b$ . Further properties of this scheme are given as follows:

- At  $\lambda = 0.5(a + b)$  the symmetry axis is defined.
- For symmetry axis, the solution for  $H^{-1}(0)$  is known or computationally simple to obtain. This point is known as the initial point of the homotopy method.
- At  $\lambda = a$ , the homotopy formulation becomes

$$H(f(x), a) = f(x) = 0. \tag{3.1}$$

This means that at  $\lambda = a$  all solutions for  $f(x)$  are located.

- Similarly, at  $\lambda = b$ , the homotopy fulfils

$$H(f(x), b) = f(x) = 0, \tag{3.2}$$

which means that at  $\lambda = b$  all solutions for  $f(x)$  are located.

- The above expressions define two symmetric branches; the left branch for  $a \leq \lambda \leq (a + b)/2$  and the right branch for  $(a + b)/2 \leq \lambda \leq b$ .
- When a given branch reaches, once again the value  $\lambda = (a + b)/2$ , the homotopy procedure stops; thus a simple stop criterion is found. In Fig. 2, the paths start at point  $A$  and stop at point  $B$ .
- The path for the inverse function  $H^{-1}(0)$  is a continuous function for  $\lambda$  in the range of  $a \leq \lambda \leq b$ .

Not only the properties shown above yield a simple and reliable stop criterion, but they also establish two solution lines that, in fact, limit the swing of the homotopy parameter variation.

Fig. 2 shows how the homotopy path starts at  $A = ((a + b)/2, x_i)$  on the symmetry axis, finds two roots at  $\lambda = a$  or  $\lambda = b$  and finishes when a new crossing through the symmetry axis at  $B = ((a + b)/2, x_f)$  is detected; it means that tracing the symmetrical branch has been completed and the stop criterion is fulfilled.

### 4 DBPH homotopy

The DBPH [16] homotopy principle rests on a very simple idea: by mirroring the homotopy path from Fig. 2, a new homotopy scheme that possesses four solution lines is generated. DBPH homotopy is defined by

$$H(f(x), \lambda) = \lambda(\lambda + a)(\lambda - a)(\lambda - 2a)(x - x_i)(x - x_f) + C(\lambda - a/2)^2 f(x)^2 = 0, \tag{4.1}$$

where  $f(x)$  is the equilibrium equation,  $x$  represents the electrical variables of the circuit,  $a$  is a constant that represents the separation between solution lines,  $x_i$  is the initial point,  $x_f$  the final point, and  $C$  an arbitrary constant.

Based on the previous, homotopy can be expressed in general way as

$$H(f(x), \lambda) = \begin{cases} f(x^*) = 0, & \text{for } \lambda = 0 \text{ and } x = x^*, \\ (x - x_i)(x - x_f) = 0, & \text{for } \lambda = a/2, \\ f(x^*) = 0, & \text{for } \lambda = a \text{ and } x = x^*, \end{cases} \tag{4.2}$$

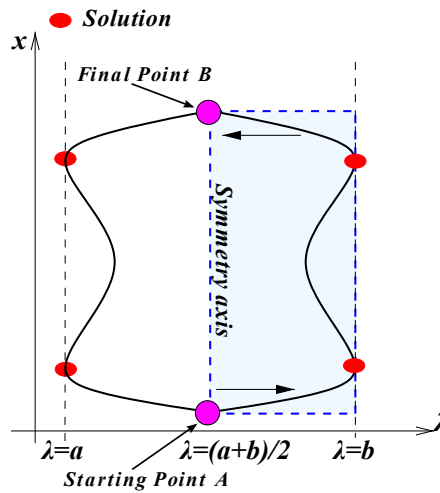


Figure 2: Homotopy with double solution line, symmetry axis, and stop criterion.

where  $x^*$  is any solution for  $f(x)$ .

The symmetry axis [16] is located at

$$\lambda = \frac{a}{2}. \tag{4.3}$$

This homotopy contains four solution lines ( $\lambda = -a, \lambda = 0, \lambda = a,$  and  $\lambda = 2a$ ) as shown in Fig. 3. Nevertheless, the two solution lines at both ends belong to unconnected open trajectories ( $\lambda_1(x)$  and  $\lambda_4(x)$ ) which are not employed for tracing purposes. Moreover, squaring function  $f(x)$  has the objective of establishing an even number of solutions which, precisely, produce the bounding and closes the homotopy path inside the middle region. In addition, aspects like symmetry axis and stop criterion were studied in detail in [16]. Furthermore, in the following subsections some novel aspects are introduced like: initial and final points and curvature radius.

### 4.1 Symmetrical branches

In order to obtain the branches for the homotopy path, (4.1) is regrouped as follows

$$H(f(x), \lambda) = \lambda(\lambda + a)(\lambda - a)(\lambda - 2a) + (\lambda - a/2)^2 J(x) = 0, \tag{4.4}$$

where

$$J(x) = \frac{Cf(x)^2}{(x - x_i)(x - x_f)}, \quad x \neq x_i, \quad x \neq x_f. \tag{4.5}$$

The symmetrical branches shown in Figure 3 can be derived solving (4.4) for  $\lambda$  at each solution line. Branches  $\lambda_2(x), \lambda_3(x)$  were reported in [16]; nevertheless, in Table 1 we present the equations for all four branches given by the DBPH homotopy:  $\lambda_1(x), \lambda_2(x), \lambda_3(x),$  and  $\lambda_4(x)$ . The third column of this table shows the limits when  $x \rightarrow x^*$ ; resulting that branches are linked to solution lines  $\lambda = [-a, 0, a, 2a]$ , respectively.

In order to trace the homotopy path, and given the fact that  $\lambda_2(x)$  and  $\lambda_3(x)$  are connected and symmetrical, only one of them may be traced to obtain the full path and end the simulation satisfactorily.  $\lambda_3(x)$  is chosen as the tracing path, which tangentially touches the solution line  $\lambda = a$ .

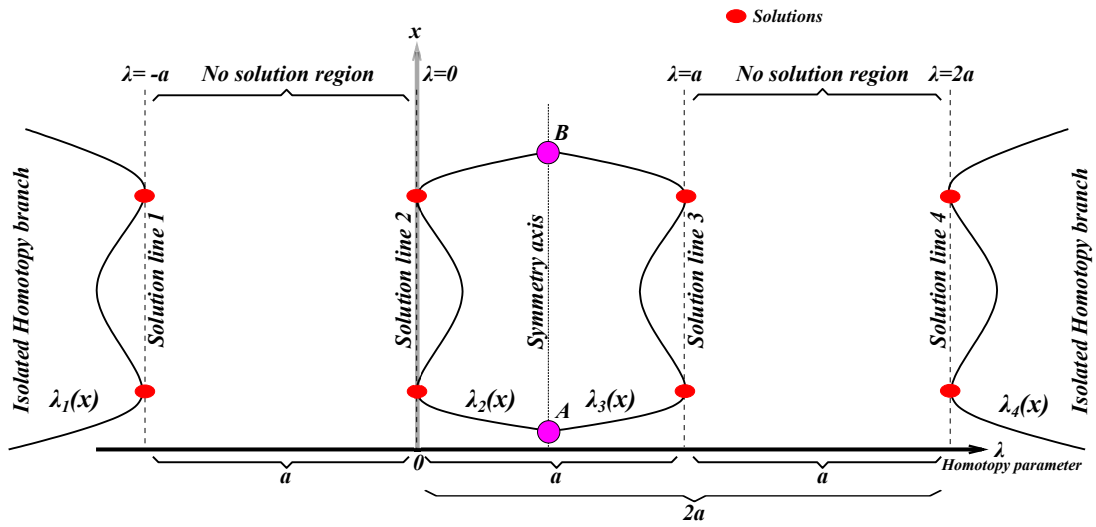


Figure 3: General scheme for DBPH homotopy.

Symmetric branches	$\lim_{x \rightarrow x^*} \lambda_k(x)$
$\lambda_1(x) = \frac{a - \sqrt{5a^2 + 2\sqrt{(J(x) - 4a^2)(J(x) - a^2)} - 2J(x)}}{2}$	$\lambda = -a$
$\lambda_2(x) = \frac{a - \sqrt{5a^2 - 2\sqrt{(J(x) - 4a^2)(J(x) - a^2)} - 2J(x)}}{2}$	$\lambda = 0$
$\lambda_3(x) = \frac{a + \sqrt{5a^2 - 2\sqrt{(J(x) - 4a^2)(J(x) - a^2)} - 2J(x)}}{2}$	$\lambda = a$
$\lambda_4(x) = \frac{a + \sqrt{5a^2 + 2\sqrt{(J(x) - 4a^2)(J(x) - a^2)} - 2J(x)}}{2}$	$\lambda = 2a$

Table 1: Symmetrical branches ( $k = 1, 2, 3$  and  $4$ ), where  $x^*$  is the solution for  $f(x)$ .

## 4.2 Initial point and final point

The point where the homotopy path starts is located at the symmetry axis, that is  $\lambda = a/2$ . Thus, by replacing  $\lambda = a/2$  in (4.1), the term that contains the equilibrium equation  $f(x)$  is cancelled, yielding

$$H(f(x), \lambda_i) = (x - x_i)(x - x_f) = 0. \quad (4.6)$$

From (4.6),  $A = (a/2, x_i)$  is the starting point for the homotopy path and  $B = (a/2, x_f)$  is the final point of the closed path, as depicted in Figure 3.

Considering  $\lambda_i = [-a, 0, a, 2a]$ , (4.1) is reduced to the original problem

$$H(f(x), \lambda_i) = f(x) = 0. \quad (4.7)$$

The stop criterion relies in the fact that the symmetry axis separates the closed homotopy path into two symmetrical branches  $\lambda_2(x)$  and  $\lambda_3(x)$ . Therefore, tracing  $\lambda_3(x)$  branch is enough to know the behaviour of the entire closed trajectory. For branch  $\lambda_3(x)$ , the solutions will be located at  $\lambda = a$  (see Table 1). As aforementioned,  $\lambda_1(x)$  and  $\lambda_4(x)$  are ignored because they exhibit open trajectories.

## 4.3 Circuit rendering

The circuit rendering the double bounded homotopy can be derived by solving  $f(x)$  from (4.1). The result is

$$f(x) = \pm \sqrt{\frac{\lambda(\lambda + a)(\lambda - a)(\lambda - 2a)(x - x_i)(x - x_f)}{(\lambda - a/2)^2}}, \quad (4.8)$$

If the nonlinear equation  $f(x)$  emanates from the analysis of a circuit, then the right hand side of (4.8) implies the existence of an artificial nonlinear resistor  $K$ ; such resistor  $K$  is due to the homotopy formulation. Furthermore,  $K$  can be a nonlinear current source or a nonlinear voltage source. That depends if the affected equation is a nodal equation or a branch relationship of a non-NA compatible element (i.e. a voltage source  $E$ ), respectively. In Fig. 4 is depicted the circuit interpretation of the homotopy map. Most of the available commercial or experimental circuit simulators use the Modified Nodal Analysis (MNA) [5] to construct the equilibrium equation  $f(x)$ . This method is based on the systematic application of Kirchhoff Laws. From MNA method, we have to point out that nonlinear resistors  $K$  are controlled by  $x$ , where  $x$  means an electrical variable (branch current or nodal voltage).

## 4.4 Curvature radius

The curvature radius of the homotopy path will be employed as a key parameter for the *qualitative* analysis about the behaviour of the paths at strategic points like solution points and turning points (see Fig. 5). It is well known that the curvature radius for a given curve in a point is the radius for a circle with curvature equivalent at that point of the curve. From Fig. 5, the equation for the curvature radius of the homotopy trajectory is given as

$$\rho = \left| \frac{(1 + (\lambda')^2)^{3/2}}{\lambda''} \right|, \quad (4.9)$$

where primes denotes derivative with respect to  $x$ .

From Fig. 5, we can establish that the slope of the homotopy path at solution points and turning points is zero ( $\lambda' = 0$ ); property that can be employed to simplify the expression for the curvature radius at those points

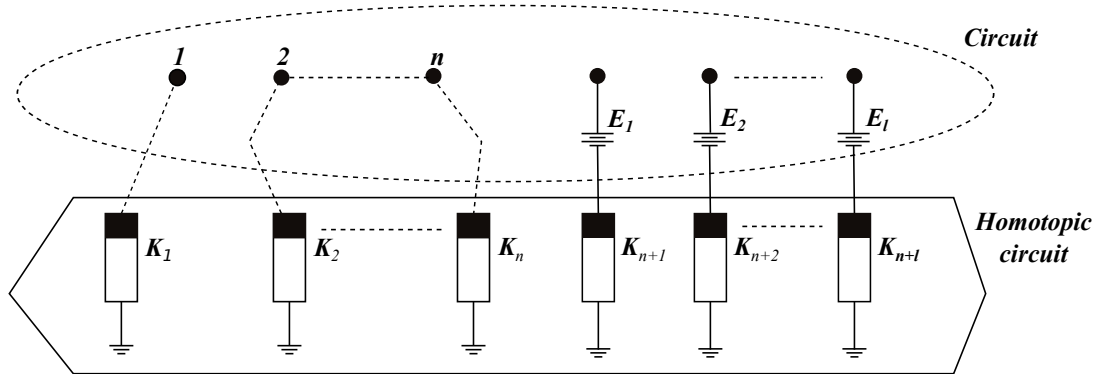


Figure 4: Circuit sketch for the double bounded Homotopy ( $n$  is the number of nodes in the circuit and  $l$  is the number of voltage sources).

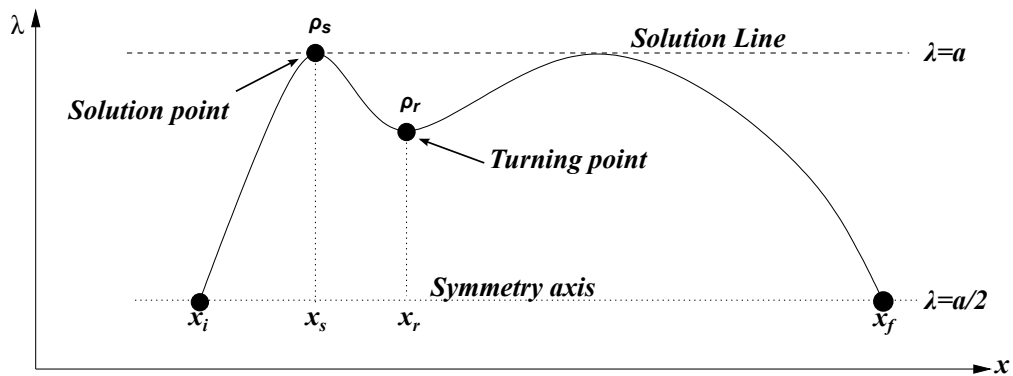


Figure 5: Solution point  $\rho_s$  and turning point  $\rho_r$ .



$$\rho = \left| \frac{1}{\lambda'} \right| \tag{4.10}$$

Now, substituting (4.5) into  $\lambda_3(x)$  from Table 1 and calculating (4.10) at solution  $x_s$ , we obtain

$$\rho_s = \left| 4 \frac{a(x-x_i)(x-x_f)}{C(f')^2} \right|_{x=x_s}, \tag{4.11}$$

where  $f'$  denotes differentiation of  $f(x)$  with respect to  $x$ .

In order to obtain (4.11), we considered the fact that equilibrium equation evaluated at the solution is zero ( $f(x_s) = 0$ ). Moreover, (4.11) reveals that the distance  $a$  between solution lines indeed affects the sharpness of the homotopy path at solution points; in such a way that the curve is more acute as solution lines approach to each other, and is flatter as solution lines separate from each other. Besides, constant  $C$  also affects the curve by flattening the trajectory when  $C$  increases.

The curvature radius at turning points is denoted as  $\rho_r$ . At these points, the derivative of  $\lambda$  with respect to  $x$  equals zero (see Fig. 5). In addition, the turning point spatially matches critical points for  $f(x)$ ; thus, the derivative of  $f(x)$  with respect to  $x$  is zero. Therefore, the simplified curvature radii  $\rho_r$  are

$$\rho_r = \left| \frac{4 \sqrt{5a^2 - 2\sqrt{4a^4 - 5Cj(x)a^2 + C^2(j(x))^2} - 2Cj(x)}}{\left( -\frac{5C\left(\frac{d^2}{dx^2}j(x)\right)a^2 + 2C^2j(x)\frac{d^2}{dx^2}j(x)}{\sqrt{4a^4 - 5Cj(x)a^2 + C^2(j(x))^2}} - 2C\frac{d^2}{dx^2}j(x) \right)} \right|_{x=x_r}, \tag{4.12}$$

where  $x_r$  is the value of  $x$  at the turning point and

$$j(x) = \frac{f(x)^2}{(x-x_i)(x-x_f)}, \quad x \neq x_i, \quad x \neq x_f. \tag{4.13}$$

In order to evaluate the qualitative behaviour of the curvature radius, we can calculate the following limits

$$\lim_{j(x) \rightarrow +\infty} |\rho_r| = \infty, \tag{4.14}$$

$$\lim_{j(x) \rightarrow +0} |\rho_r| = \left| 8 \frac{a}{C \frac{d^2}{dx^2}j(x)} \right|_{x=x_r}. \tag{4.15}$$

On one hand, the limit given in (4.14) shows that when  $j(x)$  tends to high values, the curvature radius tends to grow and the curve tends to flatten. This case indicates that homotopy path tends to stay close to solution line  $\lambda = a$ . On the other hand, the limit given in (4.15) shows that when  $j(x)$  tends to zero, the curvature radius is proportional to  $8a/C$ . This implies that small values of  $C$  can contribute to a flat trajectory around turning points.

In this section, we analysed only  $\lambda_3(x)$  because tracing this branch is enough to establish a stop criterion. Besides, from the symmetry of the homotopy trajectory, we can establish that  $\lambda_2(x)$  possesses the same properties as  $\lambda_3(x)$ .

## 5 Case study

In this section, we will show the application of DBPH homotopy to the solution of a mathematical problem and a nonlinear circuit. For both cases, we will use the numerical continuation scheme reported in [15]. This scheme is based on the following steps: euler predictor, NR corrector, variable step size, interpolation of the solutions, and NR fine tuning of the interpolated solutions.

### 5.1 Mathematical case

The first example is a nonlinear algebraic equation system

$$\begin{aligned} f_1(x_1, x_2) &= (x_2 - 1)(x_2 - 4)(x_2 - 6) + x_1 = 0, \\ f_2(x_1, x_2) &= (x_1 - 3)(x_1 - 6)(x_1 - 9) + x_2 = 0. \end{aligned} \tag{5.1}$$

Figure 6 shows the plot of the system and depicts the five possible solutions.

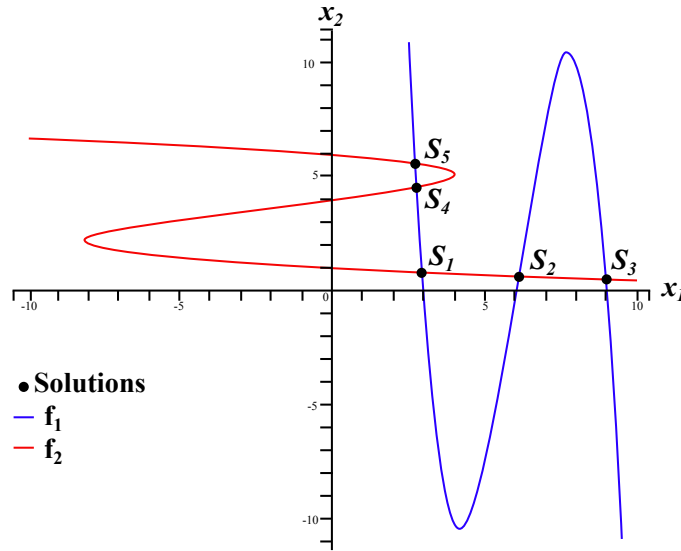


Figure 6: Equation (5.1) with five solutions.

The DBPH homotopy map (see 4.1)) can be expressed as

$$\begin{aligned} H_1(f_1, \lambda) &= \lambda(\lambda + 1)(\lambda - 1)(\lambda - 2)(x_1 - 13)(x_1 + 13) + (\lambda - 0.5)^2 f_1^2(x_1, x_2) = 0, \\ H_2(f_2, \lambda) &= \lambda(\lambda + 1)(\lambda - 1)(\lambda - 2)(x_2 - 13)(x_2 + 13) + (\lambda - 0.5)^2 f_2^2(x_1, x_2) = 0, \end{aligned} \tag{5.2}$$

considering  $a = 1$ ,  $C = 1$ ,  $x_i = -13$ , and  $x_f = 13$ .

The two solution lines of interest are located at  $\lambda = 0$  and  $\lambda = 1$ . The initial point is chosen at  $A = (x_1, x_2, \lambda) = (-13, -13, 0.5)$  and the tracing procedure of the path is started. This results in a global convergence to all solutions of the system ( $S_1, S_2, S_3, S_4$ , and  $S_5$ ) (see Fig. 7). The final point is  $B = (13, 13, 0.5)$  and the homotopy path  $\lambda_3(x)$  for each variable is shown in the  $\lambda$ - $x_1$  and  $\lambda$ - $x_2$  planes in Fig. 8(a) and 8(b), respectively. The numerical values and error of solutions are shown in Table 2. In this example, the step size of the numerical continuation is adaptive with a maximum value of  $\Delta\lambda = 0.018$ . This step size required 3932 iterations to complete the simulation.

### 5.2 Circuit with two tunnel diodes

Tunnel diodes have nonlinear behaviour, which is shown in Figure 9. As it can be seen in the figure, the tunnel diode has three sign changes of the slope. This situation combined with a load line [11] could produce up to three solutions for the tunnel diode in a circuit. The chosen model for the tunnel

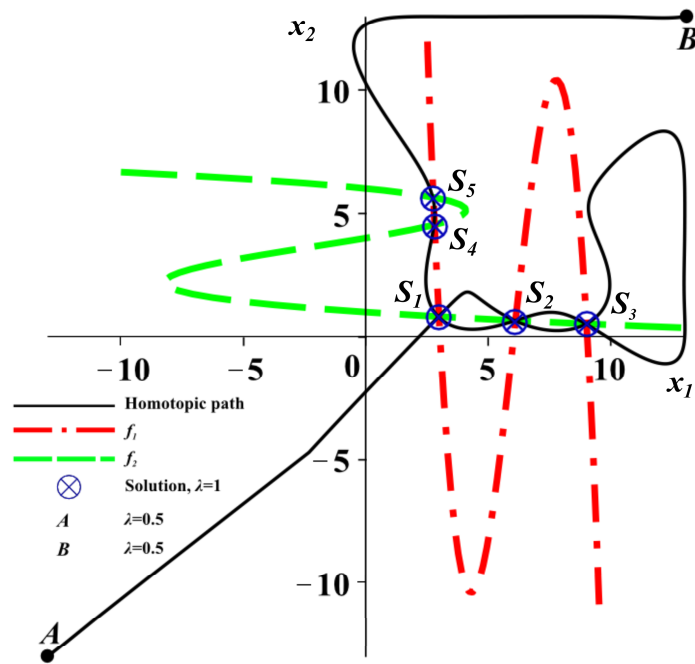
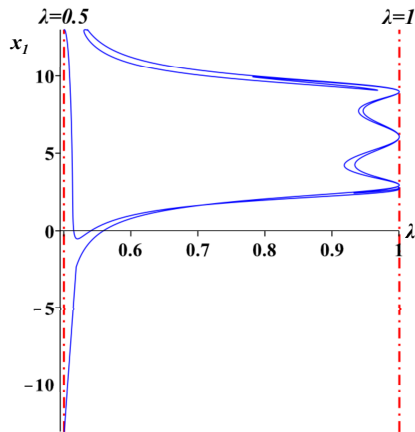


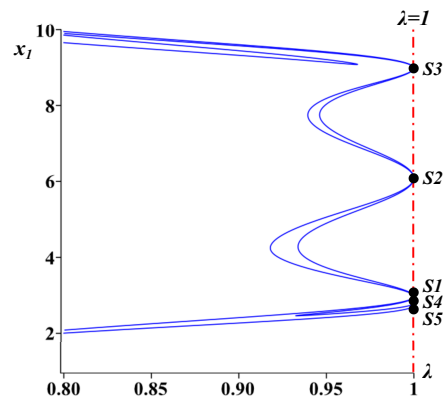
Figure 7: Projection of the homotopy trajectory of (5.2) for  $x_1-x_2$ .

Solutions	$(x_1, x_2)$	Error $\sqrt{f_1^2 + f_2^2}$
$S_1$	(2.955413610, 0.8205351687)	$2.400000000 \times 10^{-9}$
$S_2$	(6.073327848, 0.6595563496)	$2.061552813 \times 10^{-9}$
$S_3$	(8.970231771, 0.5278791756)	$1.886796226 \times 10^{-9}$
$S_4$	(2.774151458, 4.535861825)	$1.000000000 \times 10^{-9}$
$S_5$	(2.725379656, 5.642624058)	$7.071067812 \times 10^{-9}$

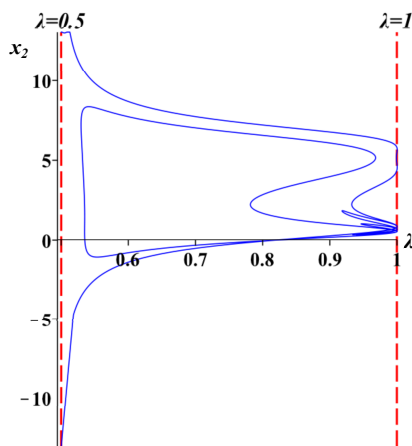
Table 2: Five solutions of (5.1).



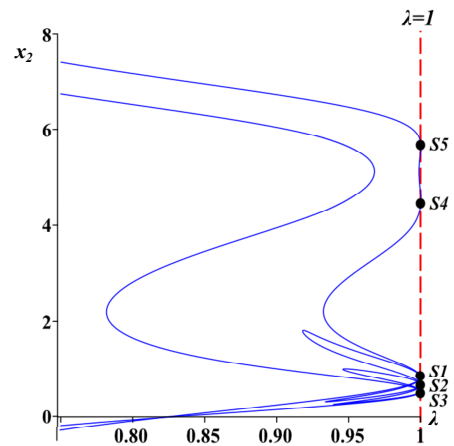
(a) Homotopy path projected over  $\lambda-x_1$ .



(b) Zoom to solutions of a).



(c) Homotopy path projected over  $\lambda-x_2$ .



(d) Zoom to solutions of c).

Figure 8: Projection of the homotopy trajectory of (5.2) for  $\lambda-x_1$  and  $\lambda-x_2$

chosen for this example contains exponential terms, and can be expressed as

$$i = I_p \left( \frac{V}{V_p} \right) \exp \left( 1 - \frac{V}{V_p} \right) + I_0 \exp \left( \frac{q}{KT} V \right). \quad (5.3)$$

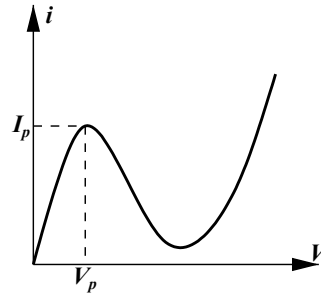


Figure 9: Model for tunnel diode.

Exponential terms can potentially cause numerical overflows, so this circuit represents a challenge for the DBPH. Therefore, this example shows a circuit (ERDD) with two tunnel diodes, one voltage source, and a resistor in series (figure 10). The value for the voltage source  $V_1$  is  $1V$  and for the resistor  $R_2$  is  $20\Omega$ . Also, both tunnel diodes ( $K_3$  y  $K_4$ ) have the same model with coefficients:  $I_p = 100 \times 10^{-3}$ ,  $V_p = 50 \times 10^{-3}$ ,  $I_0 = 1 \times 10^{-9}$ , and  $q/KT = 40$ .

The equilibrium equation is formulated from the modified nodal analysis [5]. Also, the system is reduced down to three equations eliminating the nodal voltage  $v_1$  because it is constant and equal to the voltage source. Therefore, the system variables for this system are:  $I_E$ ,  $v_2$ , and  $v_3$ . Equations are expressed as follows

$$\begin{aligned} f_1 &= \frac{1}{20} - \frac{v_2}{20} + I_E = 0, \\ f_2 &= -\frac{1}{20} + \frac{v_2}{20} + 2(v_2 - v_3) \exp(1 - 20v_2 + 20v_3) + (1E-9) \exp(40v_2 - 40v_3) = 0, \\ f_3 &= 2(v_2 - v_3) \exp(1 - 20v_2 + 20v_3) + (1E-9) \exp(40v_2 - 40v_3) \\ &\quad - 2v_3 \exp(1 - 20v_3) - (1E-9) \exp(40v_3) = 0. \end{aligned} \quad (5.4)$$

Considering  $a = 1$ ,  $C = 1$ ,  $x_i = -1$ , and  $x_f = 1$ ; the DBPH homotopy map (see 4.1)) can be expressed as

$$\begin{aligned} H_1(f_1, \lambda) &= \lambda(\lambda + 1)(\lambda - 1)(\lambda - 2)(I_E - 1)(I_E + 1) + (\lambda - 0.5)^2 f_1^2(I_E, v_2, v_3) = 0, \\ H_2(f_2, \lambda) &= \lambda(\lambda + 1)(\lambda - 1)(\lambda - 2)(v_2 - 1)(v_2 + 1) + (\lambda - 0.5)^2 f_2^2(I_E, v_2, v_3) = 0, \\ H_3(f_3, \lambda) &= \lambda(\lambda + 1)(\lambda - 1)(\lambda - 2)(v_3 - 1)(v_3 + 1) + (\lambda - 0.5)^2 f_3^2(I_E, v_2, v_3) = 0. \end{aligned} \quad (5.5)$$

Solution lines are located at  $\lambda = 0$  and  $\lambda = 1$ , and initial point for the homotopy path is selected at  $A = (I_E, v_2, v_3, \lambda) = (-1, -1, -1, 0.5)$ . Moreover, from (4.3), the symmetry axis is located at  $\lambda = 0.5$ .

As result of tracing the homotopy path, all the operating points of the circuit are located, nine in total. Fig. 11(a) shows the complete symmetric path associated to solution line  $\lambda = 1$  against nodal voltage  $v_3$ .

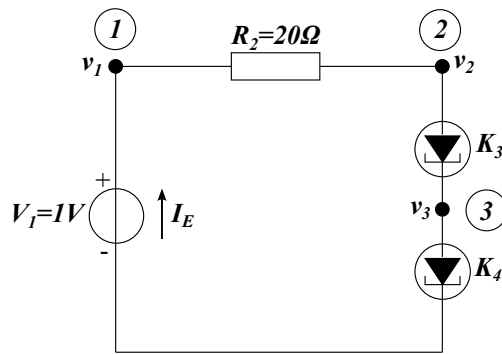


Figure 10: A circuit with two tunnel diodes.

Solution	$(I_E, v_2, v_3)$	Error $\sqrt{f_1^2 + f_2^2 + f_3^2}$
$S_1$	$(-0.0488739, 0.0225212, 0.0112606)$	$1.01581 \times 10^{-7}$
$S_2$	$(-0.0421757, 0.156486, 0.147132)$	$1.51506 \times 10^{-6}$
$S_3$	$(-0.0282802, 0.434396, 0.428548)$	$5.76360 \times 10^{-6}$
$S_4$	$(-0.0189199, 0.621602, 0.418183)$	$1.21617 \times 10^{-6}$
$S_5$	$(-0.0336137, 0.327727, 0.163863)$	$4.25969 \times 10^{-7}$
$S_6$	$(-0.0421757, 0.156486, 0.00935369)$	$3.95453 \times 10^{-7}$
$S_7$	$(-0.0282802, 0.434396, 0.00584718)$	$1.27369 \times 10^{-6}$
$S_8$	$(-0.0189199, 0.621602, 0.203419)$	$1.78300 \times 10^{-8}$
$S_9$	$(-0.00991438, 0.801712, 0.400856)$	$1.06457 \times 10^{-6}$

Table 3: Nine solutions of the tunnel circuit.

After tracing the homotopy path, all the operating points were located (a total of nine solutions (Fig. 11(a))). The final point of the path is located at  $B = (1, 1, 1, 0.5)$ . Besides, Fig. 11(b) shows a close-up in the vicinity of the solution line; it can be noticed how the path crosses over all nine solutions, presenting three sub-regions where the solutions concentrate ( $F1, F2,$  and  $F3$ ). In order to observe in detail the behaviour of the path, further enlargements of these regions are provided in Fig. 11(c), 11(d), and 11(e).

These nine solutions ( $S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8,$  and  $S_9$ ) are shown on the homotopy path. In addition, numerical values and error of these solutions are shown in Table 3.

In this example, the step size of the numerical continuation is adaptive with a maximum value of  $\Delta\lambda = 0.016$ . This step size required 343 iterations to complete the simulation.

## 6 Discussion

Numerical results from the study cases shows that DBPH homotopy has potential use to solve systems composed of nonlinear algebraic equations. In particular, equations resultant from the DC analysis of nonlinear circuits. In both study cases all solutions were located from the proposed NAEs.

In a future work, it will be necessary to research how to solve system of equations having more than three solutions and more variables.

For both case studies, the values of  $x_i$  and  $x_f$  were chosen in order to encompass all sought solutions. Furthermore, the initial point can be chosen from certain number of combinations of  $x_i$  and  $x_f$  as discussed in detail at [16]. Nevertheless, this does not mean that all possible initial points locate all solutions or even a single solution. Therefore, it is important to continue the study of this homotopy in order to choose an initial point that produce shorter simulation times or convergence to the highest number of solutions.

About the numerical continuation scheme employed to trace the homotopy trajectories of this work; we can say that each type of circuit or nonlinear problem possesses their own characteristics making it difficult to propose an appropriate value for the step size of the predictor step. Finally, we carried out a study of the curvature radius for the homotopy path at the operating points; determining that homotopy parameters directly affects the flatten performance of the homotopy path in the vicinity of the solution lines, as depicted in Figure 8 and Figure 11. Finally, future work will consist in applying the DBPH homotopy to the analysis of VLSI circuits.

## 7 Conclusions

A double-bounded homotopy scheme has been studied in detail. The proposed scheme is based on a polynomial function with four solution lines that squares the nonlinear equation to solve. The main feature of the resulting homotopy formulation resides in the fact that a simple, but still reliable, stop criterion is achieved because the homotopy path is bounded within a closed route. A set of interesting mathematical properties for the homotopy have been explained and developed. Among them, symmetry, location of the final and initial points, as well as the curvature radius at solutions and turning points were noticed. Finally, the DBPH homotopy was applied to solve a mathematical example and to find multiple DC operating points for a nonlinear circuit.

## Glossary

**DBH** Double Bounded Homotopy.

**DBPH** Double Bounded Polynomial Homotopy.

**DC** Direct Current.

**ERDD** Series circuit comprised of a voltage source, resistor, and two tunnel diodes.

**HCM** Homotopy Continuation Methods.

**MNA** Modified Nodal Analysis.

**NA** Nodal Analysis.

**NAEs** Nonlinear Algebraic Equations.

**NR** Newton-Raphson method.

**OP** Operating Point.

**VLSI** Very-large-scale of Integration.

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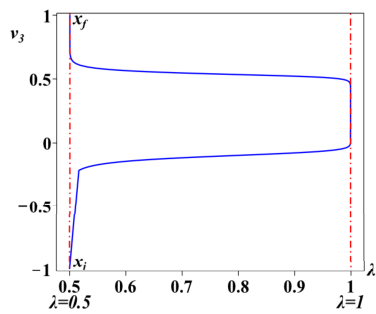
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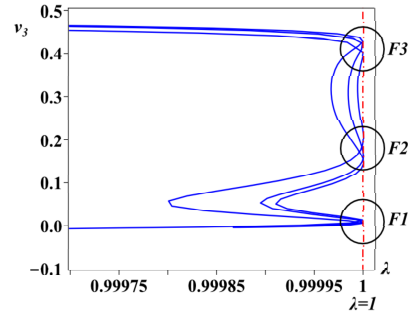
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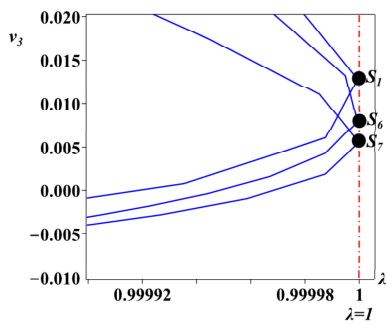
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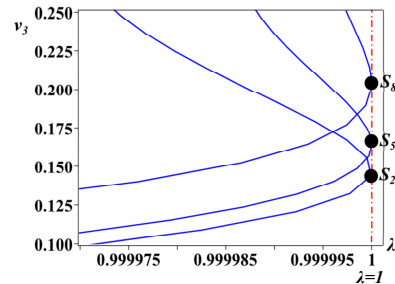
(a) Full homotopy path.



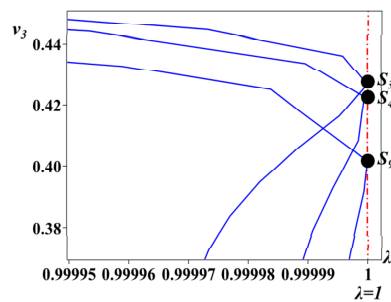
(b) Zoom to solutions region of a.



(c) Zoom to region F1.



(d) Zoom to region F2.



(e) Zoom to region F3.

Figure 11: Projection of homotopy path of (5.5) over  $\lambda-v_3$