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Article

Random attractors for Stochastic strongly damped non-autonomous wave equations with memory and multiplicative noise

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Abstract: In this paper, we study the dynamical behavior of solutions for the stochastic strongly damped wave equation with linear memory and multiplicative noise defined on $\mathbb{R}^n$. Firstly, we prove the existence and uniqueness of the mild solution of certain initial value for the above-mentioned equations. Secondly, we obtain the bounded absorbing set. Lastly, We investigate the existence of a random attractor for the random dynamical system associated with the equation by using tail estimates and the decomposition technique of solutions.

Keywords: Stochastic wave equation, linear memory, random attractor, pullback asymptotic compactness, unbounded domains.

MSC: 35B40, 35B41, 35B45, 35L05, 35R60, 58J37.

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1. Introduction

In this article, we consider the following non-autonomous strongly damped wave equation with linear memory on an unbounded domain:

$$u_{tt} - \beta\triangle u - \alpha\triangle u_t - \int_0^\infty \mu(s)\triangle\left(u(t) - u(t-s)\right)ds + f(u) = g(x,t) + cu \circ \frac{dW(x,t)}{dt}, \tag{1}$$

with initial data

$$u(\tau,x) = u_\tau(x)\ ,\ u_t(\tau,x) = u_{1,\tau}(x),\ x \in \mathbb{R}^n,\ \tau \in \mathbb{R}. \tag{2}$$

Let $\varepsilon, \alpha, \beta > 0$, c is a positive constant and $\mu(s) \leq 0$ for every $s \in \mathbb{R}^+$, where $\triangle$ is the Laplacian with respect to the variable $x \in \mathbb{R}^n$ with $n = 3$, $u = u(t,x)$ is a real function of $x \in \mathbb{R}^n$, $(n = 3)$,and $t \geq \tau$, $\tau \in \mathbb{R}$. The function $g(x,t) \in Ł_b^2(\mathbb{R}, L^2(\mathbb{R}^n))$ is time-dependent external force, and $W(x,t)$ is an independent two sided real-valued wiener processes of probability space.

Following a well-established procedure first devised by [1], we introduce a Hilbert " history " space $\mathfrak{R}_\mu = L^2_\mu(\mathbb{R}^+, H^1(\mathbb{R}^n))$ with the inner product and new variants.

$$\begin{cases} (\eta_1,\eta_2)_{\mu,1} = \displaystyle\int_0^\infty \mu(s)(\nabla\eta_1(s),\nabla\eta_2(s))ds, \\ \eta(x,t,s) = u(x,t) - u(x,t-s), \\ \eta_t = \dfrac{\partial}{\partial t}\eta\ ,\ \eta_s = \dfrac{\partial}{\partial s}\eta. \end{cases} \tag{3}$$

Then the Equation (1) can be transformed into the following system

$$\begin{cases} u_{tt} - \beta\triangle u - \alpha\triangle u_t - \displaystyle\int_0^\infty \mu(s)\triangle\eta(s)ds + f(u) = g(x,t) + cu \circ \frac{dW(x,t)}{dt}, \\ \eta_t = -\eta_s + u_t, \end{cases} \tag{4}$$

with the initial-boundary conditions

$$\begin{cases} u(\tau,x) = u_\tau(x), \\ u_t(\tau,x) = u_{1\tau}(x)\,, x \in \mathbb{R}^n,\ \tau \in\ \mathbb{R}, \\ \eta_\tau(x,\tau,s) = \eta_\tau = u_\tau(x) - u_\tau(x-s)\,, x \in \mathbb{R}^n,\ \tau \in\ \mathbb{R}, s \in\ \mathbb{R}^+. \end{cases} \tag{5}$$

The following conditions are necessary to obtain our main results.

1. concerning the memory kernel μ, it is required to satisfy the following hypotheses:

$$\begin{cases} \mu \in \mathbb{C}^1(\mathbb{R}^+) \cap \mathbb{L}^1(\mathbb{R}^+), \mu(s) \geq 0, \mu'(s) \leq 0\,, \forall s \in \mathbb{R}^+, \\ \mu'(s) + \delta\mu(s) \leq 0\,, \forall s \in \mathbb{R}^+ and\ \delta > 0, \end{cases} \tag{6}$$

and denote

$$k_0 := \int_0^\infty \mu(s)ds < \infty. \tag{7}$$

2. for the nonlinear term $f(u)$, we assume that $f \in \mathbb{C}^1(\mathbb{R})$ with $f(0) = 0$,and it satisfies the following growth conditions. There exist constant $C_1 > 0$ such that

$$|f'(u)| \leq C_1(1 + |u|^p)\,, \forall\ u \in \mathbb{R}, 0 \leq p \leq 4,\ when\ n = 3. \tag{8}$$

And there exists constants $k > 0$ and $\nu_1 >$ such that for any $\nu \in (0, \nu_1)$, there exist $C_\nu > 0$ satisfying

$$kF(u) - \nu u^2 + C_\gamma \leq uf(u),\ \forall\ u \in \mathbb{R}, \tag{9}$$

and

$$\limsup_{|u|\to\infty} \frac{f(u)}{u} \leq 0\,, \forall\ u \in\ \mathbb{R}, \tag{10}$$

where $F(s) = \int_0^s f(r)dr$.

About the time-dependent forcing $g(x,t)$ term we assume that $g(x,t) \in Ł_b^2(\mathbb{R}, L^2(\mathbb{R}^n))$, where space of translation -bounded function $Ł_b^2(\mathbb{R}, L^2(\mathbb{R}^n)) = \{g(x,t) \in Ł_{loc}^2(\mathbb{R}, L^2(\mathbb{R}^n)) : \sup_{t\in\mathbb{R}} \int_t^{t+1} (\int_{\mathbb{R}^n} |g(\cdot,r)|^2 dx) dr < \infty\}$ with the norm

$$\|g(x,t)\|^2 = \sup_{t\in\mathbb{R}} \int_t^{t+1} \int_{\mathbb{R}^n} |g(x,r)|^2 dxdr < \infty,\quad \forall r \in \mathbb{R}. \tag{11}$$

Finally, we introduce the product Hilbert space

$$E = H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times \mathfrak{R}_\mu.$$

In recent years, there have many results on the dynamics of a variety of systems related to Equation (1). The deterministic hyperbolic equations with memory have been studied to possess global attractors which, despite being subsets of an infinite-dimensional phase space, are finite-dimensional objects, see[1–12]. For instance, Borini and Pata [13] proved the existence of a Uniform attractor for a strong damping wave equation with linear memory on a bounded domain. Qiaozhen Ma,Chengkui Zhong[7] obtained the strong global attractors, and Ghidaglia, and Marzocchi [14] showed global attractors and their finite Dimension. Crauel and Flandoli [15–18] studied the random attractors for a stochastic dynamical system. Recently, many authors have established the existence of random attractors for other equations (see[13,19–34]). For Equation (1), there are fewer results and most previous authors have concentrated to the deterministic case, but there are no results of random attractors for the Equation (1).

In general, to prove the existence of random attractors for (1) in E, we must establish the pullback asymptotic compactness of solutions. Since Sobolev embedding are not compact on $\mathbb{R}^n$, we cannot get the desired asymptotic compactness directly from the regularity of solutions. We were overcome the difficulty by using uniform estimates on the tails of solutions outside a bounded ball in $\mathbb{R}^n$ and decomposing the solutions in a bounded domain as in [2,6,21,29].

The rest of the article is organized as follows. In Section 2 we recall some basic concepts related to RDS and a random attractor for the random dynamical system. In Section 3, we devote to uniform estimates and the existence of bounded absorbing sets for the solutions and pullback compactness. In Section 4, the compactness of the RDS is established by the decomposition of a solution of the random differential equation into two parts. In Section 5, we prove the asymptotic compactness of the solutions, finally existence and uniqueness of a random attractor in E.

2. Preliminaries

In this Section, we recall some basic concepts related to RDS and a random attractor for RDS in [16,17], which are important for getting our main results. Let $(\Omega, \mathcal{F}, P)$ be a probability space and (X, d) is a Polish space with the Borel σ-algebra $B(X)$.The distance between $x \in X$ and $B \subseteq X$ is denoted by $d(x, B)$. If $B \subseteq X$ and $C \subseteq X$, the Hausdorff semi-distance from B to C is denoted by $d(B, C) = \sup_{x \in B} d(x, C)$.

Definition 1. $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is called a metric dynamical system if $\theta : \mathbb{R} \times \Omega \longrightarrow \Omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable, θ_0 is the identity on Ω, $\theta_{s+t} = \theta_t \circ \theta_s$, for all s,t $\in \mathbb{R}$ and θ_0P = P for all t$\in \mathbb{R}$.

Definition 2. A mapping $\Phi(t, \tau, \omega, x) : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is called continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$, if for all $\tau \in \mathbb{R}, \omega \in \Omega$ and $t, s \in \mathbb{R}^+$, the following conditions are satisfied:

1. $\Phi(t, \tau, \omega, x) : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is a $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F}, \mathcal{B}(\mathbb{R}))$ measurable mapping
2. $\Phi(0, \tau, \omega, x)$ is identity on X.
3. $\Phi(t + s, \tau, \omega, x) = \Phi(t, \tau + s, \theta_s\omega, x) \circ \Phi(s, \tau, \omega, x)$
4. $\Phi(t, \tau, \omega, x) : X \to X$ is continuous.

Definition 3. Let 2^X be the collection of all subsets of X, set valued mapping $(\tau, \omega) \mapsto \mathcal{D}(t, \omega) : \mathbb{R} \times \Omega \mapsto 2^X$ is called measurable with respect to $\mathcal{F}$ in Ω if $\mathcal{D}(t, \ \omega)$ is a(usually closed) nonempty subset of X and the mapping $\omega \in \Omega \mapsto d(X, B(\tau, \omega))$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$ -measurable for every fixed $x \in X$ and $\tau \in \mathbb{R}$. Let $B = B(t, \omega) \in \mathcal{D}(t, \omega) : \tau \in \mathbb{R}, \omega \in \Omega$ is called a random set.

Definition 4. A random bounded set $B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ of X is called tempered with respect to $\{\theta(t)\}_{t\in\Omega}$, if for p-a.e $\omega \in \Omega$,

$$\lim_{t \mapsto \infty} e^{-\beta t} \, d(B(\theta_{-t}\omega)) = 0, \ \forall \ \beta \ > 0,$$

where

$$d(B) = \sup_{x \in B} \|x\|_X.$$

Definition 5. Let $\mathcal{D}$ be a collection of random subsets of X and $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, then K is called an absorbing set of $\Phi \in \mathcal{D}$, if for all $\tau \in \mathbb{R}, \omega \in \Omega$ and $B \in \mathcal{D}$, there exists, $T = T(\tau, \omega, B) > 0$ such that.

$$\Phi(t, \tau, \theta_{-t}\omega, B(\tau, \theta_{-t}\omega)) \subseteq K(\tau, \omega), \ \forall \ t \ \geq T$$

Definition 6. Let $\mathcal{D}$ be a collection of random subsets of X, then Φ is said to be $\mathcal{D}$ a-pullback asymptotically compact in X, if for p-a.e $\omega \in \Omega$, $\{\Phi(t_n, \theta_{-t_n}\omega \ , x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in X when $t_n \mapsto \infty$ and $x_n \in B(\theta_{-t_n}\omega)$ with $\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$.

Definition 7. Let $\mathcal{D}$ be a collection of random subsets of X and $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, then $\mathcal{A}$ is called a $\mathcal{D}$-random attractor (or $\mathcal{D}$-pullback attractor) for Φ, if the following conditions are satisfied, for all $t \in \mathbb{R}^+, \tau \in \mathbb{R}$ and $\omega \in \Omega$

1. $\mathcal{A}(\tau, \omega)$ is compact, and $\omega \mapsto d(x, \mathcal{A}(\omega))$ is measurable for every $x \in X$

2. $\mathcal{A}(\tau,\omega)$ is invariant, that is

$$\Phi(t,\tau,\omega,\mathcal{A}(\tau,\omega)) = \mathcal{A}(\tau+t,\theta_t\omega), \forall\, t \geq \tau.$$

3. $\mathcal{A}(\tau,\omega)$ attracts every set in $\mathcal{D}$, that is for every $B = \{B(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$,

$$\lim_{t \mapsto \infty} d_X(\Phi(t,\tau,\theta_{-t}\omega, B(\tau,\theta_{-t}\omega)), \mathcal{A}(\tau,\omega)) = 0.$$

Where d_X is the Hausdorff semi-distance given by

$$d_X(Y,Z) = \sup_{y\in Y} \inf_{z\in Z} \|y-z\|_X$$

for any $Y \in X$ and $Z \in X$.

Lemma 8. *Let $\mathcal{D}$ be a neighborhood-closed collection of (τ,ω)- parameterized families of nonempty subsets of X and Φ be a continuous cocycle on X over $\mathbb{R}$ and $(\Omega,\mathcal{F},P,(\theta_t)_{t\in\mathbb{R}})$. Then Φ has a pullback $\mathcal{D}$-attractor $\mathcal{A}$ in $\mathcal{D}$ if and only if Φ is pullback $\mathcal{D}$-asymptotically compact in X and Φ has a closed, $\mathcal{F}$-measurable pullback $\mathcal{D}$-absorbing set $K \in \mathcal{D}$, the unique pullback $\mathcal{D}$-attractor $\mathcal{A} = \mathcal{A}(\tau,\omega)$ is given $\mathcal{A}(\tau,\omega) = \bigcap_{r\geq 0}\overline{\bigcup_{t\geq r}\Phi(t,\tau,\theta_{-t}\omega, K(\tau,\theta_{-t}\omega))}$ $\tau \in \mathbb{R}$,$\omega \in \Omega$.*

3. Existence and uniqueness of solutions

In this section, we present the existence and uniqueness of solutions for the system (1)-(2). It is well known that the operator $A = -\triangle$ with the domain $D(A) = H^2(\mathbb{R}^n)$.

We recall some important results, let $H_0 = L^2(\mathbb{R}^n)$, $H_1 = H^1(\mathbb{R}^n)$ and $H_2 = \mathfrak{R}_\mu = L^2_\mu(\mathbb{R}^+, H^1(\mathbb{R}^n))$. And denote $H_1^* = H^{-1}(\mathbb{R}^n)$ the dual space of H_1, as usual, we identify H_0^*, the dual space of H_0. Then we get

$$\begin{cases} (u,v) = \displaystyle\int_{\mathbb{R}^n} uv dx, \|u\| = (u,u)^{\frac{1}{2}}, \forall u,v \in L^2(\mathbb{R}^n), \\ ((u,v)) = \displaystyle\int_{\mathbb{R}^n} \nabla u \nabla v dx,\ \|\nabla u\| = ((u,u))^{\frac{1}{2}},\ \forall\, u,v \in H^1(\mathbb{R}^n), \\ (\eta,\zeta)_{\mu,1} = \displaystyle\int_0^\infty \mu(s)(\nabla\eta(s),\nabla\zeta(s))ds, \\ \|\eta\|^2_{\mu,1} = (\eta,\eta)_{\mu,1} = \displaystyle\int_0^\infty \mu(s)(\nabla\eta(s),\nabla\eta(s))ds. \end{cases} \tag{12}$$

$E = E(\mathbb{R}^n) = H_0 \times H_1 \times H_2 = H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times \mathfrak{R}_\mu$, endowed with the usual norms on E,

$$\|'\|^2_E = \|\varphi\|^2_{H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\times\mathfrak{R}_\mu}.$$

Due to the Ornstein-Uhlenbeck process deducing by the Brownian motion, which holds the *Itô* differential equation

$$dz + \delta z dt = dw,\ \delta > 0 \tag{13}$$

and hence the solution is given by

$$\begin{aligned} &\theta_t\omega(s) = \omega(t+s) - \omega(t), \\ &z(\theta_t\omega) = z(t,\omega) \ = \ -\delta\int_{-\infty}^0 e^{\delta s}(\theta_t\omega)s ds, s \in \mathbb{R}, \omega \in \Omega. \end{aligned} \tag{14}$$

Where the random variable $|z(\omega)|$ is tempered and there is an invariant set $\bar{\Omega} \subseteq \Omega$ of full P measure such that $z(\theta_t\omega) = z(t,\omega)$ is continuous in t for every $\omega \in \bar{\Omega}$. This equation has a random fixed point in the sense of random dynamical systems generating a stationary a solution is known as the stationary Ornstein-Uhlenbeck process (see [16,17,29,35] for more details).

Next we need to transform the stochastic system into deterministic with a random parameter, then show that it generates a random dynamical system. In fact, we define a cocycle for problem (12)-(14). Let

$$v = \frac{du}{dt} + \varepsilon u - cuz(\theta_t\omega), \tag{15}$$

by (15) and (4), (5), we can obtain the following random evolution equation

$$\begin{cases} u_t + \varepsilon u - v = cuz(\theta_t\omega), \\ v_t + \varepsilon(\varepsilon - \alpha A)u + \beta A u - (\varepsilon - \alpha A)v + \int_0^\infty \mu(s)A\eta(s)ds \\ \quad = -f(u) + g(x,t) + cz(\theta_t\omega)\left(v + \varepsilon u - \alpha A u + cuz(\theta_t\omega)\right), \\ \eta_t + \eta_s + \varepsilon u - v = cuz(\theta_t\omega), \end{cases} \tag{16}$$

with the initial-boundary conditions

$$\begin{cases} u(\tau,x) = u_\tau(x), \\ u_t(\tau,x) = u_{1,\tau}(x), x \in \mathbb{R}^n,\ \tau \in\ \mathbb{R}, \\ \eta_\tau(x,s) = u_\tau(x) - u_\tau(x,\tau-s), x \in \mathbb{R}^n,\ \tau \in\ \mathbb{R}, s \in\ \mathbb{R}^+. \end{cases} \tag{17}$$

Which, in contrast to the stochastic differential Equation (1)-(2), can by analysis pathwise with deterministic calculus, define

$$\varphi = \begin{pmatrix} u \\ v \\ \eta \end{pmatrix},$$

$$L\varphi = \begin{pmatrix} \varepsilon u - v \\ \varepsilon(\varepsilon - \alpha A)u + \beta A u - (\varepsilon - \alpha A)v + \int_0^\infty \mu(s)A\eta(s)ds \\ \varepsilon u - v + \eta_s \end{pmatrix}$$

and

$$Q(\varphi,\omega,t) = \begin{pmatrix} cuz(\theta_t\omega) \\ cu(\varepsilon - \alpha A)z(\theta_t\omega) + c^2uz^2(\theta_t\omega) + cvz(\theta_t\omega) - f(u) + g(x,t) \\ cuz(\theta_t\omega) \end{pmatrix}$$

Then the following equation is equivalent to the system (15)-(17)

$$\begin{cases} \varphi' + L\varphi = Q(\varphi,t,\omega) \\ \varphi\tau = (u_\tau(x), u_{1\tau}(x) + \varepsilon u_\tau(x) - cu_\tau z(\theta_t\omega), \eta_\tau(x,s))^\top. \end{cases} \tag{18}$$

In line with [36], we know that -L is the infinitesimal generator of C^0 semigroup e^{-Lt} on E for $t > 0$, by the assumptions (6)-(11). It is easy to check $Q(\varphi,t,\omega) : E \to E$ is locally Lipschitz continuous with respect to φ, by the classical semigroup theory concerning the (local) existence and uniqueness solutions of evolution differential equation, we have the following theorem.

Theorem 9. *Under the condition (6)-(11) and for each $\tau \in \mathbb{R}, \omega \in \Omega$ and for any $\varphi_\tau \in E$, there exists $T > 0$ such that (18) has a unique mild function $\varphi(t,\tau,\omega,\varphi_\tau) \in C([\tau,\tau+T);E)$ and $\varphi(t)$ satisfies the integral equation*

$$\varphi(t,\tau,\omega,\varphi_\tau) = e^{-L(t-\tau)}\varphi_\tau(\omega) + \int_\tau^t e^{-L(t-r)}Q(\varphi,r,\omega)dr, \tag{3.8}$$

$\varphi(t,\tau,\omega,\varphi_\tau)$ is jointly continuous into t and measurable in ω.

From Theorem 9, we know that for P-a.s. each $\omega \in \Omega$, the following results hold for all $T > 0$

1. if $\varphi_\tau(\omega) \in E$ then $\varphi(t,\omega,\varphi_\tau(\tau)) \in C([\tau,\tau+T);E)$,
2. $\varphi(t,\tau,\omega,\varphi_\tau)$ is jointly continuous into t and measurable in $\varphi_\tau(\omega)$,
3. the solution mapping of (18) satisfies the properties of continuous cocycle.

We notice that a unique solution $\varphi(t,\tau,\omega,\varphi_\tau)$ of (18) can define a continuous random dynamical system over $\mathbb{R}$ and $(\Omega,\mathcal{F},P,(\theta_t)_{t\in\mathbb{R}})$. Hence the solution mapping

$$\begin{aligned}&\bar{\Phi}(t,\omega):\mathbb{R}\times\Omega\times E\mapsto E, t\geq\tau,\\&\varphi(\tau,\omega)=(u_\tau,v_\tau,\eta_\tau)^\top\mapsto(u(t,\omega),v(t,\omega),\eta(t,\omega))^\top=\varphi(t,\omega),\end{aligned}\tag{19}$$

generates a random dynamical system. Moreover,

$$\Phi(t,\omega):\varphi(\tau,\omega)+(0,\varepsilon z(\theta_\tau\omega),0)^\top\mapsto\varphi(t,\omega)+(0,\varepsilon z(\theta_t\omega),0)^\top.\tag{20}$$

We also define the following transformation:

$$\psi_1=u,\ \psi_2=u_t+\varepsilon u.\tag{21}$$

Similar to (18), we get that

$$\begin{cases}\psi'+H\psi=Q(\psi,t,\omega)\\ \psi_\tau=(u_\tau,v_\tau,\eta_\tau)^\top=(u_\tau,u_{1\tau}+\varepsilon u_\tau,\eta_\tau)^\top,\end{cases}\tag{22}$$

where

$$\psi=\begin{pmatrix}u\\v\\\eta\end{pmatrix},$$

$$H\psi=\begin{pmatrix}\varepsilon u-v\\\varepsilon(\varepsilon-\alpha A)u+\beta Au-(\varepsilon-\alpha A)v+\eta\\\varepsilon u-v+\eta_s\end{pmatrix}$$

and

$$Q(\psi,\omega,t)=\begin{pmatrix}0\\cvz(\theta_t\omega)-f(u)+q(x,t)\\0\end{pmatrix}$$

We introduce the isomorphism $T_\epsilon Y=(u,u_t,\eta)^\top$, $Y=(u,v,\eta)^\top\in E$ which has inverse isomorphism $T_{-\epsilon}Y=(u,v-\varepsilon u,\eta)^\top$, it follows that (θ,ψ) with mapping

$$\Psi=T_\epsilon\Phi(t,\omega)T_{-\epsilon}=\Psi(t,\omega)\tag{23}$$

is a random dynamical system from a above discussion, we show that the two RDS are equivalent.

4. Random absorbing set

In this section, we will show the existence of a random absorbing set for the RDS $\varphi(t,\tau,\omega,\varphi_\tau(\omega)), t\geq 0$ in the space E. Let $\varphi=(u,v,\eta)^\top=(u,u_t+\varepsilon u-cuz(\theta_t\omega),\eta)^\top$, where ε is chosen as

$$\varepsilon=\frac{\alpha\lambda_1+\beta_1}{4+2(\alpha\lambda_1+\beta_1)\alpha+\beta_2^2/\lambda_1}.\tag{24}$$

Lemma 10. *For any $\varphi=(u,v,\eta)^\top\in E$ we have*

$$(L\varphi,\varphi)_E\geq\frac{\varepsilon}{2}\left(\|u\|_1^2+\|v\|^2\right)+\frac{\alpha}{2}\|v\|^2+\frac{\varepsilon}{4}\|\eta\|_{\mu,1}^2.\tag{25}$$

Proof. This is easily obtained by simple computation. □

Lemma 11. *Assume that (6)-(11) hold, then for each $\tau\in\mathbb{R},\omega\in\Omega$, there exists tempered random absorbing ball $B_0(\tau,\omega)=\{\varphi\in E:\|\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\|_E\leq M(\tau,\omega)\}$, $B_E(M(\tau,\omega))\in\mathcal{D}(E)$, such that for any* *set $B \in \mathcal{D}(E)$, there exists $T_B = T_B(\tau, \omega, B) > 0$,$\tau \in \mathbb{R}, \omega \in \Omega, B \in \mathcal{D}$, so as $\forall t \geq T_B$ and $\varphi_{\tau-t}(\theta_{-\tau}\omega) \in B(\tau - t, \theta_{-t}\omega)$, the solution of a system (4.7) satisfies*

$$\|\varphi(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega))\|_E^2 \leq M^2(\tau, \omega) \tag{26}$$

that is

$$\Phi(\tau, \tau - t, \theta_{-\tau}\omega, B(\tau - t, \theta_{-t}\omega)) \subseteq B_0(\tau, \omega)), \ \forall\, t \geq \tau. \tag{27}$$

Proof. For any $\tau \in \mathbb{R}, \omega \in \Omega, t \geq \tau$, let $\varphi(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega)) = (u_\tau, v_\tau, \eta_\tau) \in E$, be a mild solution of (18) with initial value $\varphi_{\tau-t}$.
Taking the inner product $(\cdot, \cdot)_E$ of (18) with $\varphi(\tau)$, we find that

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|_E^2 + \frac{\varepsilon}{2}(\|u\|_1^2 + \|v\|^2) + \frac{\alpha}{2}\|v\|^2 + \frac{\varepsilon}{4}\|\eta\|_{\mu,1}^2 \leq (Q(\varphi, t, \omega), \varphi). \tag{28}$$

Let us estimate the right hand side of (28)

$$\begin{aligned}(Q(\varphi, \omega, t), \varphi) \ &= \ ((cuz(\theta_t\omega), u)) + (cu(\varepsilon - \alpha A)z(\theta_t\omega) + c^2uz^2(\theta_t\omega) + cvz(\theta_t\omega) \\ &\quad -f(u) + g(x, t), v) + (cuz(\theta_t\omega), \eta)_{\mu,1}.\end{aligned} \tag{29}$$

By the Cauchy-Schwartz inequality, we find that

$$((cuz(\theta_t\omega), u)) \leq |c||z(\theta_t\omega)|\|\nabla u\|^2 \leq |c||z(\theta_t\omega)|\|u\|_1^2, \tag{30}$$

$$\varepsilon(cuz(\theta_t\omega), v) \leq \varepsilon|c||z(\theta_t\omega)|\|u\|\|v\| \leq \frac{\varepsilon|c||z(\theta_t\omega)|}{2\sqrt{\lambda_0}}(\|u\|_1^2 + \|v\|^2), \tag{31}$$

$$(c^2uz^2(\theta_t\omega), v) \leq \frac{1}{\sqrt{\lambda_0}}|c|^2|z(\theta_t\omega)|^2\|u\|_1\|v\| \leq \frac{2|c|^4|z(\theta_t\omega)|^4}{\varepsilon\lambda_0} + \frac{\varepsilon}{8}(\|u\|_1^2 + \|v\|^2), \tag{32}$$

$$(cvz(\theta_t\omega), v) \leq \frac{|c||z(\theta_t\omega)|}{2}\|v\|^2, \tag{33}$$

$$\alpha(c\nabla uz(\theta_t\omega), \nabla v) \leq \alpha|c||z(\theta_t\omega)|\|u\|\|v\| \leq \frac{\alpha\sqrt{\lambda_1}|c||z(\theta_t\omega)|}{2}(\|u\|_1^2 + \|v\|^2), \tag{34}$$

$$(cuz(\theta_t\omega), \eta)_{\mu,1} \leq |c||z(\theta_t\omega)|\|u\|_1\|\eta\|_{\mu,1} \leq \frac{|c||z(\theta_t\omega)|}{2}(\|u\|_1^2 + \|\eta\|_{\mu,1}^2), \tag{35}$$

$$(g(x, t), v) \leq \|g(x, t)\| \ \|v\| \leq \frac{2}{(4\alpha + \varepsilon)}\|g(x, t)\|^2 + \frac{(4\alpha + \varepsilon)}{8}\|v\|^2, \tag{36}$$

here we estimate nonlinear term (29), by (6)-(10) and the Hölder inequality, we get that

$$(f(u), v) = \left(f(u), \frac{du}{dt} + \varepsilon u - cuz(\theta_t\omega)\right) \geq \frac{d}{dt}\int_{\mathbb{R}^n} F(u)dx + \varepsilon\,(f(u), u) - (f(u), cuz(\theta_t\omega)) \tag{37}$$

Due to (4),(6), (1-8) and poincarè inequality, there exists positive constant μ_1, μ_2 such that

$$(f(u), u) - k\tilde{F}(u) + \mu_1\|u\|_1 + \mu_2 \geq 0, \tag{38}$$

it follows from (10) for each given $\mu_3 > 0$

$$(f(u), u) \geq \mu_2\|u\|_1 + \mu_3, \tag{39}$$

$$(f(u), v) \leq \frac{d}{dt}\tilde{F}(u) + \varepsilon k\tilde{F}(u) - (\mu_1\varepsilon + \mu_3 cz(\theta_t\omega))\|u\|_1 - \varepsilon\mu_2 - c_{\mu_3}|c||z(\theta_t\omega)|, \tag{40}$$

where $\tilde{F}(u)=\int_{\mathbb{R}^n}F(u)dx$. Collecting (30)-(40) and (29), we show that

$$\begin{aligned}(Q(\varphi,\omega,t),\varphi) \quad &\leq \quad -\frac{d}{dt}\tilde{F}(u)-\varepsilon k\tilde{F}(u)+\frac{\varepsilon}{4}(\|u\|_1^2+\|v\|^2)+\mu_4|c|\;|z(\theta_t\omega)|(\|u\|_1^2+\|v\|^2+\|\eta\|_{\mu,1}^2)\\ &+\varepsilon\mu_2+c_{\mu_3}|c|\;|z(\theta_t\omega)|^2+\frac{2|c|^4|z(\theta_t\omega)|^4}{\varepsilon\lambda_0}+\frac{2}{(4\alpha+\varepsilon)}\|g(x,t)\|^2+\frac{\alpha}{2}\|v\|^2, \qquad (41)\end{aligned}$$

where μ_4 depends on $\mu_3,\frac{\varepsilon}{2\sqrt{\lambda_0}},\frac{\alpha\sqrt{\lambda_0}}{2}$. Then substituting all together into (28) yield

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|u\|_1^2+\|v\|^2+\|\eta\|_{\mu,1}^2+2\tilde{F}(u))\leq-\left(\frac{\varepsilon}{4}-\mu_4|c||z(\theta_t\omega)|)\right)(\|u\|_1^2+\|v\|^2+\|\eta\|_{\mu,1}^2)-\varepsilon k\tilde{F}(u)\\ &+\varepsilon\mu_2+\mu_3|c|\;|z(\theta_t\omega)|^2+\frac{2|c|^4|z(\theta_t\omega)|^4}{\varepsilon\lambda_0}+\frac{2}{(4\alpha+\varepsilon)}\|g(x,t)\|^2. \qquad (42)\end{aligned}$$

Since $\sigma=\min[\frac{\varepsilon}{4},\frac{\varepsilon k}{2}]$ and $\|\varphi\|^2=(\|u\|_1^2+\|v\|^2+\|\eta\|_{\mu,1}^2)$, then we have the following equivalent system

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\varphi\|_E^2+2\tilde{F}(u)) \quad &\leq \quad (\mu_4|c||z(\theta_t\omega)|-\sigma)\left(\|\varphi\|_E^2+\tilde{F}(u)\right)+\varepsilon\mu_2+c_{\mu_3}|c|\;|z(\theta_t\omega)|^2\\ &+\frac{2|c|^4|z(\theta_t\omega)|^4}{\varepsilon\lambda_0}+\varepsilon\lambda_0+\frac{1}{(2\alpha+\varepsilon)}\|g(x,t)\|^2. \qquad (43)\end{aligned}$$

Let $\Gamma(\omega)=\sigma-\mu_4|c||z(\theta_t\omega)|$ and $|z(\theta_t\omega)|$ is tempered, by (13) and (14), we can choose the following inequality

$$\varrho(t,\omega)=\beta(1+|c|^2|z(\theta_t\omega)|^2+\frac{2|c|^4|z(\theta_t\omega)|^4}{\varepsilon\lambda_0}+\|g(x,t)\|^2),$$

where $\beta>0$ depend only on $\mu_2,c_{\mu_3},C,\varepsilon,\alpha,\lambda_0$. By applying Gronwall's inequality to (43) over $[\tau-t,\tau]$ and replacing ω to $\theta_{-\tau}\omega$, we have

$$\begin{aligned}&\|\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\|_E^2\leq\left(\|\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\|_E^2+2\tilde{F}(u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\right)\\ &\leq\left(\|\varphi_{\tau-t}(\theta_{-\tau}\omega)\|_E^2+2\tilde{F}(u_{\tau-t})\right)e^{-2\Gamma t}+\int_{-t}^{0}\varrho(r-\tau,\theta_{r-\tau}\omega)e^{2\Gamma(r-\tau,\omega)}dr. \qquad (44)\end{aligned}$$

Suppose that

$$\begin{aligned}y\left(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega)\right) \quad &= \quad \left(\|\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\|_E^2+2\tilde{F}(u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}))\right)\\ &\geq \quad \|\varphi(\tau,\tau-t,\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\|_E^2\\ &\geq \quad 0 \qquad (45)\end{aligned}$$

Using (8) and Young inequality, the embedding theorem, we have for any bounded set B of E, where $\sup_{\varphi\in B}\|\varphi\|_E\leq M(\tau,\omega)$, if $\varphi(\tau)\in B$, then

$$\begin{aligned}2\tilde{F}(u) \quad &\leq \quad k\int_{\mathbb{R}^n}(f(u)+1)udx\leq k\int_{\mathbb{R}^n}(f(u)udx+C_4\int_{\mathbb{R}^n}udx\\ &\leq \quad k\|u\|^2+k\|u\|_{H^1}^{P+1}\leq k\|u\|^2+k\|u\|_{H^1}^{P+1}\leq\mu_6r_1(\omega). \qquad (46)\end{aligned}$$

For any set $\{B(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$, $\varphi_\tau=(u_\tau(x),u_{1,\tau}(x)+\varepsilon u_\tau(x)-cu_\tau z(\theta_t\omega))^\top\in\{B(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}(E)$.
We have

$$\lim_{t\to\infty}\sup\left(\|\varphi_\tau(\theta_{-\tau}\omega)\|_E^2+2C_8\left(\|u_\tau\|^2+\|u_\tau\|_{H^1}^{P+2}\right)\right)e^{-2\sigma t}=0,$$

$$\int_{-t}^{0}\varrho(r-\tau,\theta_{r-\tau}\omega)e^{2\Gamma(r-\tau,\omega)}dr<\infty. \qquad (47)$$

When $g(x,t)$ is only satisfied (11) which is a tempered random variable, then by (46)-(47), there exists $B_0(\omega) = \{\varphi \in E : \|\varphi_\tau(\theta_{-\tau}\omega)\|_E \leq M^2(\tau,\omega)\}$ is closed measurable absorbing ball in D(E) and $T = T(\tau, B, \omega) > 0$ such that $\varphi(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}) = \varphi_\tau \in B_0(\omega)$ satisfy the following result p-a.s $\omega \in \Omega$

$$\|\varphi(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t})\|_E^2 \leq M^2(\tau,\omega).$$

□

Next, we conduct uniform estimates on the tail parts of the solutions for large space variables when the time is sufficiently large in order to prove the pullback asymptotic compactness of the cocycle associated with equation (15) on the unbounded domain $\mathbb{R}^n$. We first, choose a smooth function ρ defined on $\mathbb{R}^+$ such that $0 \leq \rho(s) \leq 1$ for all $s \in \mathbb{R}$ and

$$\rho(s) = \begin{cases} 0\,, \ \forall\, 0 < |s| \leq 1, \\ 1\,, \ \forall\, |s| \geq 2. \end{cases} \tag{48}$$

Then there exist constants μ_7 and μ_8 such that $|\rho'(s)| \leq \mu_1, |\rho''(s)| \leq \mu_2$ for any $s \in \mathbb{R}$, given $r \geq 1$, denote by $\mathbb{H}_r = \{x \in \mathbb{R}^n : |x| < r\}$ and $\{\mathbb{R}^n \backslash \mathbb{H}_r\}$ the complement of $\mathbb{H}_r$. To prove asymptotic compactness of the random dynamical system we prove the following Lemma.

Lemma 12. *Under conditions (6)-(11) and B=$\{B(\tau,\omega)\}_{\tau\in\mathbb{R},\omega\in\Omega} \in \mathcal{D}$ and $\varphi_\tau(\omega) \in B$. Then there exist $\tilde{T} = \tilde{T}(\tau, B, \omega) > 0$ and $R = R(\tau, B, \omega) > 1$ so that the solution $\varphi(t, \tau, \theta_{-t}\omega, \varphi_\tau(\theta_{-t}\omega))$ of (15) satisfies for P-a.e $\omega \in \Omega$, $\forall\, t \geq \tilde{T}, r \geq R$*

$$\|\varphi\,(t, \tau, \theta_{-t}\omega, \varphi_\tau(\theta_{-t}\omega))\|^2_{E(\mathbb{R}^n\backslash\mathbb{H}_r)} \leq \epsilon\,. \tag{49}$$

Proof. Multiplying the second term of (15) with $\rho\left[\frac{|x|^2}{r^2}\right] v$ in $L^2(\mathbb{R}^n)$ and integrating over $\mathbb{R}^n$, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \rho\left[\frac{|x|^2}{r^2}\right]|v|^2\,dx \;=\;& \varepsilon\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]|v|^2\,dx + \alpha\int_{\mathbb{R}^n}(\nabla v)\rho\left[\frac{|x|^2}{r^2}\right]vdx - |\varepsilon|^2\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]uvdx \\
&+(\alpha\varepsilon-\beta)\int_{\mathbb{R}^n}(\triangle u)\,\rho\left[\frac{|x|^2}{r^2}\right]vdx + \int_{\mathbb{R}^n}\int_0^\infty \mu(s)(\triangle\eta(s))\rho\left[\frac{|x|^2}{r^2}\right]vdsdx \\
&+3|c|\,|z(\theta_t\omega)|\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]uvdx - |c|^2\,|z(\theta_t\omega)|^2\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]uvdx \\
&-|c|\,|z(\theta_t\omega)|\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]|v|^2\,dx + \alpha|c|\,|z(\theta_t\omega)|\int_{\mathbb{R}^n}(\triangle u)\,\rho\left[\frac{|x|^2}{r^2}\right]vdx \\
&+\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]g(x,t)vdx - \int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]f(u)vdx.
\end{aligned} \tag{50}$$

In order to estimate the left hand side, we must substituting v in the first term of (15), then we obtain the following results

$$\begin{aligned}
\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]uvdx \;=\;& \int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]u\left[\frac{du}{dt}+\varepsilon u - cuz(\theta_t\omega)\right]dx \\
\leq\;& \int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]\left[\frac{1}{2}\frac{d}{dt}|u|^2 + \varepsilon|u|^2 - |c||u|^2||z(\theta_t\omega)|\right]dx,
\end{aligned} \tag{51}$$

$$\begin{aligned}
\int_{\mathbb{R}^n}(-\triangle u)\,\rho\left[\frac{|x|^2}{r^2}\right]vdx \;=\;& \int_{\mathbb{R}^n}(\nabla u)\nabla\left[\rho\left[\frac{|x|^2}{r^2}\right]\left[\frac{du}{dt}+\varepsilon u - cuz(\theta_t\omega)\right]\right]dx \\
=\;& \int_{\mathbb{R}^n}\nabla u\left[\frac{2x}{r^2}\rho'\left[\frac{|x|^2}{r^2}\right]v\right]dx + \int_{\mathbb{R}^n}(\nabla u)\left[\rho\left[\frac{|x|^2}{r^2}\right]\nabla\left[\frac{1}{2}\frac{du}{dt}+\varepsilon u - cuz(\theta_t\omega)\right]\right]dx \\
\leq\;& \frac{\sqrt{2}}{r}\mu_7\left(\|\nabla u\|^2 + \|v\|^2\right) + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]|\nabla u|^2\,dx \\
&+\varepsilon\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]|\nabla u|^2dx - |c|\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]|\nabla u|^2|\nabla z(\theta_t\omega)|dx.
\end{aligned} \tag{52}$$

By second Equation (4), we have

$$\int_{\mathbb{R}^n}\int_0^\infty \mu(s)(-\triangle\eta(s))\rho\left[\frac{|x|^2}{r^2}\right]vdsdx \le \int_{\mathbb{R}^n}\int_0^\infty \mu(s)\nabla\eta(s)\nabla\left[\rho\left[\frac{|x|^2}{r^2}\right]\left[\frac{du}{dt}+\varepsilon u - cz(\theta_t\omega)\right]\right]dsdx$$
$$= \int_{r<|x|<2\sqrt{2}r}\frac{2x}{r^2}\mu_1\int_0^\infty \mu(s)\nabla\eta(s)vdsdx + \int_{\mathbb{R}^n}\int_0^\infty \mu(s)\rho\left[\frac{|x|^2}{r^2}\right]|\nabla\eta(s)||\nabla u_t|dsdx$$
$$+\varepsilon\int_{\mathbb{R}^n}\int_0^\infty \mu(s)\rho\left[\frac{|x|^2}{r^2}\right]|\nabla\eta(s)||\nabla u|dsdx - c\int_{\mathbb{R}^n}\int_0^\infty \mu(s)\rho\left[\frac{|x|^2}{r^2}\right]|\nabla\eta(s)||\nabla u||\nabla z(\theta_t\omega)|dsdx. \quad (53)$$

Integrating by parts, assumption (4),(6) and (7) and Young inequality, we can show that

$$\int_{\mathbb{R}^n}\int_0^\infty \mu(s)\rho\left[\frac{|x|^2}{r^2}\right]\nabla\eta(s)\nabla u_t dsdx \le \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]|\eta(s)|^2_{\mu,1}dx + \frac{\delta}{2}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]|\eta(s)|^2_{\mu,1}dx, \quad (54)$$

then

$$\varepsilon\int_{\mathbb{R}^n}\int_0^\infty \mu(s)\rho\left[\frac{|x|^2}{r^2}\right]\nabla\eta(s)\nabla udsdx \le \frac{\delta}{2}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]|\eta(s)|^2_{\mu,1}dx + \frac{2m_0\varepsilon^2}{\delta}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]|\nabla u|^2dx, \quad (55)$$

and

$$c\int_{\mathbb{R}^n}\int_0^\infty \mu(s)\rho\left[\frac{|x|^2}{r^2}\right](\nabla\eta(s))(\nabla u)z(\theta_t\omega)dsdx \le \frac{\delta|c||z(\theta_t\omega)|}{2}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]|\eta(s)|^2_{\mu,1}dx$$
$$+\frac{2m_0|c||z(\theta_t\omega)|}{\delta}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]|\nabla u|^2dx, \quad (56)$$

by (54), (55) and (56), it follows that

$$\int_{\mathbb{R}^n}\int_0^\infty \mu(s)(-\triangle\eta(s))\rho\left[\frac{|x|^2}{r^2}\right]vdsdx \le \frac{\sqrt{2}}{r}\mu_7(\|\nabla\eta\|^2_\mu+\|v\|^2) + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]|\eta(s)|^2_{\mu,1}dx$$
$$+\delta\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]|\eta(s)|^2_{\mu,1}dx + \frac{2m_0\varepsilon^2}{\delta}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]|\nabla u|^2dx - \frac{\delta|c||z(\theta_t\omega)|}{2}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]|\eta(s)|^2_{\mu,1}dx$$
$$-\frac{2m_0|c||z(\theta_t\omega)|}{\delta}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]|\nabla u|^2dx, \quad (57)$$

and

$$\begin{aligned}\int_{\mathbb{R}^n}(-\triangle v)\rho\left[\frac{|x|^2}{r^2}\right]vdx &= \int_{\mathbb{R}^n}\nabla v\nabla\left[\rho\left[\frac{|x|^2}{r^2}\right]v\right]dx\\ &\le \int_{\mathbb{R}^n}\nabla v\left(\frac{2x}{r^2}\rho'\left[[\frac{|x|^2}{r^2}\right]v\right)dx + \int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]|\nabla v|^2dx\\ &\le \int_{r<|x|<\sqrt{2}r}\frac{2x}{r^2}\mu_7|\nabla v||v|dx + \int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]|\nabla v|^2dx\\ &\le \frac{\sqrt{2}}{r}\mu_7(\|\nabla v\|^2+\|v\|^2) + \int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]|\nabla v|^2dx. \end{aligned} \quad (58)$$

For the nonlinear term, according to (9), (10), (40) and applying Young inequality, after detailed computations, we obtain

$$-\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]f(u)vdx \ge -\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]F(u)dx - \varepsilon k\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]F(u)dx$$
$$+\frac{(\mu_1\varepsilon+\mu_3|c||z(\theta_t\omega)|)}{2\lambda_0}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]|\nabla u|^2dx + \left(\mu_2\varepsilon - c_{\mu_3}|c||z(\theta_t\omega)|\right)\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]dx. \quad (59)$$

By the Cauchy-Schwartz inequality, the Young inequality and $\|\nabla v\|^2 \ge \lambda_1\|v\|^2$, we deduce that

$$\int_{\mathbb{R}^n} \rho\left[\frac{|x|^2}{r^2}\right] g(x,t)v dx \leq \int_{\mathbb{R}^n} \rho\left[\frac{|x|^2}{r^2}\right] \frac{|g(x,t)|^2}{4\alpha\lambda_1} dx + \alpha \int_{\mathbb{R}^n} \rho\left[\frac{|x|^2}{r^2}\right] |\nabla v|^2 dx, \tag{60}$$

$$\begin{aligned} & cz(\theta_t\omega)\left(3\varepsilon - cz(\theta_t\omega)\right) \int_{\mathbb{R}^n} \rho\left[\frac{|x|^2}{r^2}\right] uv dx \leq cz(\theta_t\omega)\left(3\varepsilon + cz(\theta_t\omega)\right) \int_{\mathbb{R}^n} \rho\left[\frac{|x|^2}{r^2}\right] |u||v| dx \\ & \leq \frac{1}{2}\left(3\varepsilon|c||z(\theta_t\omega)| + c^2|z(\theta_t\omega)|^2\right) \int_{\mathbb{R}^n} \rho\left[\frac{|x|^2}{r^2}\right]\left[|u|^2 + |v|^2\right] dx, \end{aligned} \tag{61}$$

$$\begin{aligned} & \alpha cz(\theta_t\omega) \int_{\mathbb{R}^n} (-\triangle u)\rho\left[\frac{|x|^2}{r^2}\right] v dx \leq \alpha|c||z(\theta_t\omega)| \int_{\mathbb{R}^n} (\nabla u)\nabla\left[\rho\left[\frac{|x|^2}{r^2}\right] v\right] dx \\ = \; & \alpha|c||z(\theta_t\omega)| \int_{\mathbb{R}^n} \frac{2|x|}{r^2}\rho'\left[\frac{|x|^2}{r^2}\right] |\nabla u| v dx + \alpha|c||z(\theta_t\omega)| \int_{\mathbb{R}^n} \rho\left[\frac{|x|^2}{r^2}\right] |\nabla u||\nabla v| dx \\ & \leq \alpha|c||z(\theta_t\omega)| \frac{\sqrt{2}}{r}\mu_7(\|\nabla u\|^2 + \|v\|^2) + \frac{\alpha|c||z(\theta_t\omega)|}{2} \int_{\mathbb{R}^n} \rho\left[\frac{|x|^2}{r^2}\right] |v|^2 dx \\ & + \frac{\alpha\lambda_1|c||z(\theta_t\omega)|}{2} \int_{\mathbb{R}^n} \rho\left[\frac{|x|^2}{r^2}\right] |\nabla u|^2 dx. \end{aligned} \tag{62}$$

Combining with (50)-(62) and (50), we see that

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^n} \rho\left[\frac{|x|^2}{r^2}\right]\left(|v|^2 + \varepsilon^2|u|^2 + (\beta - \alpha\varepsilon)|\nabla u|^2 + |\eta(s)|^2_{\mu,1} + 2\tilde{F}(u)\right) dx \\ & \leq \varepsilon \int_{\mathbb{R}^n} \rho\left[\frac{|x|^2}{r^2}\right]\left[|v|^2 - \varepsilon^2|u|^2 - (\beta - \alpha\varepsilon - \frac{\varepsilon\,(2m_0 - \mu_1\delta)}{\delta})|\nabla u|^2\right] dx \\ & -\varepsilon k \int_{\mathbb{R}^n} \rho\left[\frac{|x|^2}{r^2}\right] \tilde{F}(u) dx - \delta \int_{\mathbb{R}^n} \rho\left[\frac{|x|^2}{r^2}\right] |\eta(s)|^2_{\mu,1} dx \\ & +|c||z(\theta_t\omega)| \int_{\mathbb{R}^n} \rho\left[\frac{|x|^2}{r^2}\right]\left[-\frac{(\alpha+1)}{2}|v|^2 + \varepsilon^2|u|^2\right] \\ & +|c||z(\theta_t\omega)| \int_{\mathbb{R}^n} \rho\left[\frac{|x|^2}{r^2}\right]\left[(\beta - \alpha\varepsilon - \frac{\alpha\lambda_1\varepsilon}{2} - \frac{\varepsilon\,(2m_0 - \mu_3\delta)}{\delta})|\nabla u|^2 + \delta|\eta(s)|^2_{\mu,1}\right] dx \\ & +\frac{\sqrt{2}}{r}\mu_7\left[\alpha(\|\nabla v\|^2 + \|v\|^2) + (\beta - \alpha\varepsilon)(\|\nabla u\|^2 + \|v\|^2)\right] \\ & +\frac{\sqrt{2}}{r}\mu_7\left[\alpha|c||z(\theta_t\omega)(\|\nabla u\|^2 + \|v\|^2 + \|\eta\|^2_{\mu,1} + \|v\|^2\right] \\ & +\frac{1}{2}\left(3\varepsilon|c||z(\theta_t\omega)| + c^2|z(\theta_t\omega)|^2\right) \int_{\mathbb{R}^n} \rho\left[\frac{|x|^2}{r^2}\right]\left[|u|^2 + |v|^2\right] dx \\ & + \int_{\mathbb{R}^n} \rho\left[\frac{|x|^2}{r^2}\right] \frac{|g(x,t)|^2}{4\alpha\lambda_1} dx + \left(\mu_2\varepsilon - c_{\mu_3}|c||z(\theta_t\omega)|\right) \int_{\mathbb{R}^n} \rho\left[\frac{|x|^2}{r^2}\right] dx. \end{aligned}$$

Letting (provided ε is small enough)

$$\begin{cases} \sigma = \min[\varepsilon, \varepsilon k, \delta], \dfrac{(\alpha+1)}{2} \geq \dfrac{\alpha}{4}, \\ \beta - \alpha\varepsilon - \dfrac{\varepsilon\,(2m_0 - \mu_1\delta)}{\delta} \geq \beta - \alpha\varepsilon, \\ \beta - \alpha\varepsilon - \dfrac{\alpha\lambda_1\varepsilon}{2} - \dfrac{\varepsilon\,(2m_0 - \mu_3\delta)}{\delta} \geq \beta - \alpha\varepsilon, \\ \sigma_2 = \min \dfrac{\sqrt{2}}{k}\mu_1(\beta - \alpha\varepsilon, \alpha, \delta), \\ \Upsilon(t,\omega) = \dfrac{|g(x,t)|^2}{4\alpha\lambda_1} dx + \left(\mu_2\varepsilon - \mu_3|c||z(\theta_t\omega)|\right). \end{cases} \tag{63}$$

By all the above inequality, we can write that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]\left[|v|^2+\varepsilon^2|u|^2+(\beta-\alpha\varepsilon)|\nabla u|^2+|\eta(s)|^2_{\mu,1}+2\tilde{F}(u)\right]dx\\
&\leq -\sigma+|c||z(\theta_t\omega)\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]\left[|v|^2+\varepsilon^2|u|^2+(\beta-\alpha\varepsilon)|\nabla u|^2+\tilde{F}(u)+|\eta(s)|^2_{\mu,1}\right]dx\\
&+\sigma_2\left[\|\nabla v\|^2+\|\nabla u\|^2+\|\eta\|^2_{\mu,1}+\|v\|^2\right]\\
&+\frac{1}{2}\left(3\varepsilon|c||z(\theta_t\omega)|+c^2|z(\theta_t\omega)|^2\right)\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]\left[|u|^2+|v|^2\right]dx+\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]\Upsilon(t,\omega)dx. \qquad (64)
\end{aligned}$$

Setting

$$\begin{aligned}
\mathbb{X}(t,\tau,\omega,\mathbb{X}_\tau(\omega)) &= |v(t,\tau,\omega,v_\tau(\omega))|^2+\varepsilon^2|u(t,\tau,\omega,u_\tau(\omega))|^2\\
&+(\beta-\alpha\varepsilon)|\nabla u(t,\tau,\omega,u_\tau(\omega))|^2+|\eta(t,\tau,\omega,\eta_\tau(\omega),s)|^2_{\mu,1}, \qquad (65)
\end{aligned}$$

then it follows that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]\left[|\mathbb{X}(t,\tau,\omega,\mathbb{X}_\tau(\omega))|+\tilde{F}(u)\right]dx\\
&\leq -2\left[\sigma-\frac{1}{2}\left(3\varepsilon|c||z(\theta_t\omega)|+c^2|z(\theta_t\omega)|^2\right)\right]\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]\left[|\mathbb{X}(t,\tau,\omega,\mathbb{X}_\tau(\omega))|+\tilde{F}(u)\right]dx\\
&+\sigma_2\left[\|\nabla v\|^2+\|\nabla u\|^2+\|\eta\|^2_{\mu,1}+\|v\|^2\right]+\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]\Upsilon(t,\omega)dx. \qquad (66)
\end{aligned}$$

Integrating (66) over $[\tau,t]$, we find that, for all $t\geq\tau$

$$\begin{aligned}
&\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]\left[|\mathbb{X}(t,\tau,\omega,\mathbb{X}_\tau)|^2_E+2\tilde{F}(u(t,\tau,\omega,u_\tau)\right]dx\\
&\leq e^{2\sigma_1(t-\tau)}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]\left[|\mathbb{X}_\tau|^2_E+\tilde{F}(u_\tau))\right]dx+\sigma_2\int_\tau^t e^{2\sigma_1(r-t)}\left(\|\nabla v(s,\tau,\omega,v_\tau)\|^2+\|\nabla u(s,\tau,\omega,u_\tau)\|^2\right.\\
&\left.+\|\eta(t,\tau,s,\omega,\eta_\tau)\|^2_{\mu,1}+\|v(s,\tau,\omega,v_\tau)\|^2\right)dr+C\int_\tau^t e^{2\sigma_1(r-t)}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]\Upsilon(r,\theta_r\omega)dxdr, \qquad (67)
\end{aligned}$$

where $\sigma_1=\sigma-\frac{1}{2}\left(3\varepsilon|c||z(\theta_t\omega)|+c^2|z(\theta_t\omega)|^2\right)$. By replacing ω *by* $\theta_{-t}\omega$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]\left[|\mathbb{X}(t,\tau,\theta_{-t}\omega,\mathbb{X}_\tau(\theta_{-t}\omega))|^2_E+2\tilde{F}(u(t,\tau,\theta_{-t}\omega,u_\tau)\right]dx\\
&\leq e^{2\sigma_1(t-\tau)}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]\left[|\mathbb{X}_\tau(\theta_{-t}\omega)|^2_E+\tilde{F}(u_\tau))\right]dx+C\int_{\tau-t}^0 e^{2\sigma_1(r-t)}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]\Upsilon(r-t,\theta_{r-t}\omega)dxdr\\
&+\sigma_2\int_{\tau-t}^0 e^{2\sigma_1(r-t)}(\|\nabla v(r,\tau,\theta_{r-t}\omega,v_\tau)\|^2+\|\nabla u(r,\tau,\theta_{r-t}\omega,u_\tau)\|^2\\
&+\|\eta(r,\tau,s,\theta_{r-t}\omega,\eta_\tau)\|^2_{\mu,1}+\|v(r,\tau,\theta_{r-t}\omega,v_\tau)\|^2)dr. \qquad (68)
\end{aligned}$$

Since $\mathbb{X}_\tau=(u_\tau,v_\tau,\eta_\tau)^\top\in B(\tau,\theta_{-t}\omega),B\in\mathcal{D}$ is tempered, by (46)-(47), we find that the first term on the right-hand side of (68) goes to zero as $t\to-\infty$. Hence, there exist $T_1(\tau,B,\omega)>0$ and $\mathbb{R}_1=\mathbb{R}_1(\tau,\omega,B)$ such that for all such that $t\geq T_1$

$$\lim_{r\to-\infty}e^{-\sigma r}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]\left[|\mathbb{X}_\tau(\theta_{-t}\omega)|^2_E+\tilde{F}(u_\tau)\right]\leq 2\epsilon, \qquad (69)$$

by condition (11), (13)-(14) and Lemma 10, there are $T_2=T_2(\tau,B,\omega)>0$ and $\mathbb{R}_1=\mathbb{R}_1(\tau,\omega)\geq 1$ such that for all $t\geq T_2$ and $R\geq\mathbb{R}_1$

$$C\int_{\tau-t}^0 e^{2\sigma_1(r-t)}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]\Upsilon(r-t,\theta_{r-t}\omega)dxdr\leq\epsilon. \qquad (70)$$

By Lemma 10, there are $T_3 = T_3(\tau, B, \omega) > 0$ and $\mathbb{R}_2 = \mathbb{R}_2(\tau,\omega) \geq 1$ such that for all $t \geq T_3$ and $r \geq \mathbb{R}_2$

$$\begin{aligned} \sigma_2 \int_{\tau-t}^{0} e^{2\sigma_1(r-t)} \Big(&\|\nabla v(r-t,\tau,\theta_{r-t}\omega, v_\tau)\|^2 + \|\nabla u(r-t,\tau,\theta_{r-t}\omega, v_\tau)\|^2 \\ &+\|\eta(r-t,\tau,s,\theta_{r-t}\omega,\eta_\tau)\|^2_{\mu,1} + \|v(r-t,\tau,\theta_{r-t}\omega,v_\tau)\|^2 \Big)\, dr \leq \epsilon. \end{aligned} \tag{71}$$

By letting

$$\begin{cases} \tilde{T} = \max\{T_1, T_2, T_3\}, \\ \mathbb{R} = \{\mathbb{R}_1, \mathbb{R}_2\}, \end{cases} \tag{72}$$

then, from (69)-(71), it follows that

$$\|\mathbb{X}(t,\tau,\theta_{-t}\omega, \mathbb{X}_\tau(\theta_{-t}\omega))\|^2_{E(\mathbb{R}^n\setminus\mathbb{H}_k)} \leq 4\epsilon. \tag{73}$$

□

5. Decomposition of equations

In order to obtain regularity estimates later, we decompose the Equation (4) by decomposing the nonlinear term. At first, we will give the following decomposition on nonlinearity $f = f_0 + f_1$, where $f_0, f_1 \in \mathbb{C}^1$ satisfy the following conditions for some proper constant: there is a constant $C > 0$ such that

$$\begin{cases} |f_0(s)| \leq C(|s| + |s|^5),\ \forall\, s \in\ \mathbb{R}, \\ s f_0(s) \geq 0\,, \\ \exists k_0 \geq 1,\ \vartheta_1 \geq 0 \textit{ such that } \forall\, \vartheta \in (0,\vartheta_1], \\ \exists\, c_\vartheta \in \mathbb{R},\ k_0 F_0(s) + \vartheta s^2 - c_\vartheta \leq s f_0(s),\ \forall\, s \in \mathbb{R} \end{cases} \tag{74}$$

and

$$\begin{cases} |f_1'(s)| \leq C(1 + |s|^p),\ \forall\, s \in\ \mathbb{R}, 0 < p \leq 4, \\ 3F_1(s) - C \leq s f_2(s), \\ -\dfrac{\lambda}{8} s^2 - C \leq F_1(s),\ \forall\, s \in \mathbb{R} \end{cases} \tag{75}$$

where
$F_i(s) = \int_0^s f_i(r)dr, i = 0, 1.$
We decompose the solution $\varphi = (u, v, \eta^t)$ into the two parts

$$\varphi = \varphi_1 + \varphi_2$$

where $\varphi_1 = (\tilde{u},\ \tilde{v}, \xi)$,$\varphi_2 = (\bar{u}, \bar{v}, \zeta)$ solves the following equation, respectively,

$$\begin{cases} \tilde{u}_{tt} - \beta\triangle\tilde{u} - \alpha\triangle\tilde{u}_t - \displaystyle\int_0^\infty \mu(s)\triangle\xi^t(s)ds + f_0(\tilde{u}) = c\tilde{u}z(\theta_t\omega), \\ \xi_t = -\xi_s + \tilde{u}_t, \\ \varphi_{1,\tau} = (\tilde{u}_\tau, \tilde{v}_\tau, \xi_\tau), \end{cases} \tag{76}$$

and

$$\begin{cases} \bar{u}_{tt} - \beta\triangle\bar{u} - \alpha\triangle\bar{u}_t - \displaystyle\int_0^\infty \mu(s)\triangle\zeta^t(s)ds + f(u) - f_0(\tilde{u}) = g(x,t) + c\bar{u}z(\theta_t\omega), \\ \zeta_t = -\zeta_s + \bar{u}_t, \\ \varphi_{2,\tau} = (\bar{u}_\tau, \bar{v}_\tau, \zeta_\tau). \end{cases} \tag{77}$$

To prove the existence of a compact random attractor for the Random Dynamical System Φ, we get the solutions of systems (76) and (77) similar to solution of a system (25), which one decays exponentially and another are bounded in higher regular space. In order to get the regularity estimate, we will prove some a priori estimates for the solutions of systems (76) on $\mathbb{R}^n \times [\tau, \infty]$

Let $\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}) = \Phi_\varepsilon(t,\tau,\omega)\varphi(\tau,\omega)$ be the solution of (15)-(17) or (18) with $\varphi(\tau,\omega)\in B_0$, set $\varphi=\varphi_1+\varphi_2$ are the basis of absorbing set Φ, suppose that $T_1 = T_1(\tau,B_0,\omega)$

$$\begin{aligned} B_1(\tau,\omega) &= \sqcup_{t\geq T_1}\varphi\left(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}\right) \\ &= \varphi_\tau \in B_1(\omega)\subseteq B_0(\omega),\ \forall\, t\geq T_1, \end{aligned} \tag{78}$$

for any $\tau\in\mathbb{R},\omega\in\Omega$, where $T_1 = T(\tau,B_0,\omega)>0$ is the pullback absorbing time in Lemma 11, then it holds $B_1(\tau,\omega)\subseteq B_0(\omega)$ such that

$$\begin{aligned} \Phi(\tau,\tau-t,\theta_{-t}\omega,B_1(\tau-t,\theta_{-t}\omega) &= \varphi(\tau,\tau-t,\theta_{-\tau}\omega,B_1(\tau-t,\theta_{-t}\omega)) \\ &\subseteq B_1(\tau,\omega)\subseteq B_0(\omega),\ \forall\, t\geq 0. \end{aligned} \tag{79}$$

Lemma 13. *Assume that (74) hold. For any $\tau\in\mathbb{R},\omega\in\Omega,t\geq 0$, there exists $M_0(\omega)>0$ such that $\varphi_1(r) = \varphi_1(r,\tau-t,\omega,\varphi_{\tau-t})$ a solution of the system (76) satisfies.*

$$\|\varphi_1(r,\tau-t,\omega,\varphi_{\tau-t})\|_E^2 \leq M_0(\omega), r\geq \tau-t. \tag{80}$$

Proof. Let $\varphi_1 = (\tilde{u},\tilde{v},\xi)^\top = (\tilde{u},\tilde{u}_t+\varepsilon\tilde{u},\xi)^\top$ be a solution of system (76), then it follows that

$$\begin{cases} \varphi_1' + \tilde{H}\varphi_1 = \tilde{F}(\varphi_1), \\ \varphi_{1,\tau} = (\tilde{u}_\tau,\tilde{v}_\tau,\xi_\tau)^\top = (\tilde{u}_\tau(x),\tilde{u}_{1\tau}(x)+\varepsilon\tilde{u}_\tau(x),\xi_\tau(x,\tau,s))^\top. \end{cases} \tag{81}$$

Taking the inner product of (81) in $L^2(\mathbb{R}^n)$ with φ_1 in E, we show that

$$\frac{1}{2}\frac{d}{dt}\|\varphi_1\|_E^2 + (\tilde{H}\varphi_1,\varphi_1) + (\tilde{F}(\varphi_1),\varphi_1) = 0, \tag{82}$$

by Lemma 10 we have

$$(\tilde{H}\varphi_1,\varphi_1) \geq \frac{\varepsilon}{2}\left(\|\tilde{u}\|_1^2 + \|\tilde{v}\|^2\right) + \frac{\alpha}{2}\|\tilde{v}\|^2 + \frac{\varepsilon}{4}\|\xi\|_{\mu,1}^2,$$

where ε satisfy (29). Now we estimate the third term of (82) such that

$$(\tilde{F}(\varphi_1),\varphi_1) = (f_0(\tilde{u}),\tilde{u}_t+\varepsilon\tilde{u}) = \frac{d}{dt}\tilde{F}_0(\tilde{u}) + \varepsilon\int_{\mathbb{R}^n} f_0(\tilde{u})\tilde{u}dx. \tag{83}$$

According to $(74)_2$ and $(74)_3$, we get

$$\begin{aligned} &F_0(\tilde{u})\geq 0\,,\ f_0(\tilde{u})\tilde{u}\geq 0, \\ &\tfrac{d}{dt}\tilde{F}_0(\tilde{u}) + \varepsilon\textstyle\int_{\mathbb{R}^n} f_1(\tilde{u})\tilde{u}dx \geq \tfrac{d}{dt}\tilde{F}_0(\tilde{u}) + k_0\varepsilon F_0(\tilde{u}) + \varepsilon\vartheta\|\tilde{u}\|^2 - \varepsilon C_\vartheta. \end{aligned} \tag{84}$$

Thus, combining with (81), (82) and (80), it follows that

$$\frac{d}{dt}\left(\|\varphi_1\|_E^2 + 2\tilde{F}_0(\tilde{u})\right) + 2\tilde{\sigma}\left(\|\varphi_1\|_E^2 + 2\tilde{F}_0(\tilde{u})\right) \leq \rho, \tag{85}$$

where $\rho=\varepsilon c_\vartheta$ and $\tilde{\sigma} = \min(\frac{\varepsilon}{2},\frac{\alpha}{2},\frac{\varepsilon}{4},k_0\varepsilon)$

$$\|\varphi_1\|_E^2 + 2\tilde{F}_0(\tilde{u}) \geq \|\varphi_1\|_E^2 \geq 0, \tag{86}$$

hence

$$\begin{aligned} \varphi_{1,\tau-t} &= (\varphi_{\tau-t}(\theta_{-\tau}\omega) + cuz(\theta_{-\tau}\omega))^\top \\ &\leq (M(\omega) + cuz(\theta_t\omega)) = \tilde{M}(\omega) \in B_0(\theta_{-t}\omega). \end{aligned} \tag{87}$$

Together $(74)_1$, (85) and (87), using Gronwall's inequality over $[r,\tau-t]$, such that, by definition of $B_0(\omega)$ and Lemma 11,

$$\|\varphi_1(r,\tau-t,\omega,\varphi_{1(\tau-t)})\|_E \leq M_0(\omega). \tag{88}$$

□

Lemma 14. *For any $\tau \in \mathbb{R}, \omega \in \Omega$ and $t > 0$, there exist positive constant $\sigma_1 \geq 0$, such that $\varphi_1(r) \in B, r \in \mathbb{R}, \omega \in \Omega, B \in \mathcal{D}$, the solution of system (76) satisfies*

$$\|\varphi_1(r, \tau - t, \omega, \varphi_{1,\tau-t}\|_E^2 \leq \bar{M}^2(\omega), \ \forall r \geq \tau - t. \tag{89}$$

Proof. Let $\varphi_1 = (\tilde{u}, \tilde{v}, \tilde{\zeta})^\top = (\tilde{u}, \tilde{u}_t + \varepsilon\tilde{u}, \tilde{\zeta})^\top$ be a solution of (76) similar to Lemma 74. Then from (74), there exists $\tilde{\sigma}(\omega) \geq 0$ and $\bar{\varrho}(\omega)$ such that

$$F_0(\tilde{u}) \geq 0, f_0(\tilde{u})\tilde{u} \geq 0, \forall \, \tilde{u} \in \mathbb{R},$$

next due to $(74)_1$ and for every $u^1 \in H^1(\mathbb{R}^n)$, by embedding theorem $H^1(\mathbb{R}^n) \subset L^6(\mathbb{R}^n) \subset L^4(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ and (81), we conclude

$$0 \leq \tilde{F}_0(\tilde{u}) \leq \int_{\mathbb{R}^n} F_0(u^1)dx \leq C(\|\tilde{u}\|^2 + \|\tilde{u}\|_{L^6}^6) \leq \bar{\varrho}(\omega)\|\tilde{u}\|_1^2, \quad \tilde{\sigma}\|\tilde{u}\|_1^2 \geq \frac{\tilde{\sigma}}{\bar{\varrho}(\omega)}\tilde{F}_0(\tilde{u}), \ \forall \, \tilde{u} \in \mathbb{R}, \tag{90}$$

due to (81) and (90), we can obtain the following result,

$$\frac{d}{dt}(\|\varphi_1\|_E^2 + 2\tilde{F}_0(\tilde{u})) + 2\tilde{\sigma}\|\varphi_1\|_E^2 + \frac{\tilde{\sigma}}{2\bar{\varrho}(\omega)}\tilde{F}_0(\tilde{u}) \leq \rho. \tag{91}$$

take $\sigma_1(\omega) = \min \ [\tilde{\sigma}, \frac{\tilde{\sigma}}{2\bar{\varrho}(\omega)}]$. By Gronwall's inequality to (91) over $[r, \tau - t]$ and replacing ω to $\theta_{-r}\omega$, we find

$$\begin{aligned} \|\varphi_1(r, \tau - t, \theta_{-r}\omega, \varphi_{1,\tau-t}(\theta_{-r}\omega)\|_E^2 &\leq \left(\|\varphi_{1,\tau-t}\|_E^2 + \tilde{F}_0(\tilde{u}_{\tau-t})\right) e^{2\sigma_1(\omega)(r+t-\tau)} + \rho\int_{\tau-t}^{r} e^{-2\sigma_1(s,\omega)}ds \\ &\leq \left(\tilde{M}(\omega) + \tilde{F}_0(\tilde{u}_{\tau-t})\right) e^{2\sigma_1(\omega)(r+t-\tau)} + \rho\int_{\tau-t}^{r} e^{-2\sigma_1(s,\omega)}ds, \end{aligned} \tag{92}$$

by $(74)_1$, we get the following estimate

$$\tilde{F}_0(\tilde{u}) = \int_{\mathbb{R}^n} F_0(u^1)dx \leq C(\|\tilde{u}\|^2 + \|\tilde{u}\|_{L^6}^6) \leq C_f\|\tilde{u}\|_{H^1}^6 \leq C_p\tilde{M}^6(\omega), \ \forall \, \tilde{u} \in \mathbb{R}. \tag{93}$$

Thus, collecting all (87) and (92)-(93), we arrive at (89), where

$$\bar{M}^2(\omega) = \left(\tilde{M}(\omega) + C_p\tilde{M}^6(\omega)\right) e^{2\sigma_1(\omega)(r+t-\tau)} + \rho\int_{\tau-t}^{r} e^{-2\sigma_1(s,\omega)}ds.$$

□

Lemma 15. *Under the conditions of (6)-(11), (74)-(75). For any $(\tau, r) \in \mathbb{R}, \omega \in \Omega$, there exists random variable radius $\varrho_2(\tau, \omega) > 0$ such that solution of the system (77) satisfies the following estimates, for all $t \geq r, r \geq \tau - t$,*

$$\left\|A^{\frac{\nu}{2}}\varphi_2(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t})\right\|_E^2 \leq \left(\left\|A^{\frac{1+\nu}{2}}\bar{u}\right\|^2 + \left\|A^{\frac{\nu}{2}}\bar{v}\right\|^2 + \left\|A^{\frac{\nu}{2}}\zeta\right\|_{\mu,1}^2\right) \leq \varrho_2^2(\tau, \omega), \tag{94}$$

where

$$\nu = min\{\frac{1}{4}, \frac{5-p}{2}\}, \ \forall \, 0 \leq p \leq 4. \tag{95}$$

Proof. Let $\varphi_2 = (\bar{u}, \bar{v}, \zeta)^\top = (\bar{u}, \bar{u}_t + \varepsilon\bar{u} - c\bar{u}z(\theta_t\omega), \zeta)^\top$, then the system (77) is equivalent to the following system with initial data

$$\begin{cases} \bar{u}_t + \varepsilon\bar{u} - \bar{v} = c\bar{u}z(\theta_t\omega), \\ \dfrac{d\bar{v}}{dt} + \varepsilon(\varepsilon - \alpha A)\bar{u} + \beta A\bar{u} - (\varepsilon - \alpha A)\bar{v} + \displaystyle\int_0^\infty \mu(s)A\zeta^t(s)ds \\ \quad = -(f(u) - f_0(\tilde{u}) + g(x,t) + cz(\theta_t\omega)(2\varepsilon\bar{u} - \alpha A\bar{u} + \bar{u}_t), \\ \zeta_t + \zeta_s + \varepsilon\bar{u} - \bar{v} = c\bar{u}z(\theta_t\omega), \end{cases} \tag{96}$$

thus, we can rewritten (96), by the following equation

$$\begin{cases} \varphi_2' + \bar{H}\varphi_2 = \bar{F}_2(\varphi_2, \omega, t), \\ \varphi_{2,\tau} = (\bar{u}_\tau, \bar{v}_\tau, \zeta_\tau)^\top, \end{cases} \tag{97}$$

where

$$\bar{H}\varphi_2 = \begin{pmatrix} \varepsilon\bar{u} - \bar{v} \\ \varepsilon(\varepsilon - \alpha A)\bar{u} + \beta A\bar{u} - (\varepsilon - \alpha A)\bar{v} + \zeta \\ \varepsilon\bar{u} - \bar{v} + \zeta_s \end{pmatrix},$$

and

$$\bar{F}_2(\varphi_2, \omega, t) = \begin{pmatrix} c\bar{u}z(\theta_t\omega) \\ c\bar{u}(\varepsilon - \alpha A)z(\theta_t\omega) + c^2\bar{u}z^2(\theta_t\omega) + c\bar{v}z(\theta_t\omega) - (f(u) - f_0(\tilde{u})) + g(x,t) \\ c\bar{u}z(\theta_t\omega) \end{pmatrix}. \tag{98}$$

Taking scalar product of (97) with $A^\nu\varphi_2(r)$, then positively

$$(\varphi_2', A^\nu\varphi_2) + (\bar{H}\varphi_2, A^\nu\varphi_2) = (\bar{F}_2(\varphi_2, \omega, t), A^\nu\varphi_2). \tag{99}$$

According to (95) and Lemma 10, we find

$$(\bar{H}\varphi_2, A^\nu\varphi_2) = \frac{\varepsilon}{2}\left(\left\|A^{\frac{1+\nu}{2}}\bar{u}\right\|^2 + \left\|A^{\frac{\nu}{2}}\bar{v}\right\|^2\right) + \frac{\alpha}{2}\left\|A^{\frac{\nu}{2}}\bar{v}\right\|^2 + \frac{\varepsilon}{4}\left\|A^{\frac{\nu}{2}}\zeta\right\|^2_{\mu,1}, \tag{100}$$

next, we will estimate the right-hand side of (99), yield

$$\begin{aligned} (\bar{F}_2(\varphi_2, \omega, t), A^\nu\varphi_2) \quad = \quad & ((c\bar{u}z(\theta_t\omega), A^\nu\bar{u})) + (c\bar{u}(\varepsilon - \alpha A)z(\theta_t\omega) + c^2\bar{u}z^2(\theta_t\omega) + c\bar{v}z(\theta_t\omega) - (f(u) - f_0(\tilde{u})) \\ & + g(x,t), A^\nu\bar{v}) + (c\bar{u}z(\theta_t\omega), A^\nu\zeta)_{\mu,1}. \end{aligned} \tag{101}$$

From (30)-(36) and (97) one by one, we get,

$$((c\bar{u}z(\theta_t\omega), A^\nu\bar{u})) \leq |c||z(\theta_t\omega)|\|A^{\frac{\nu+1}{2}}\bar{u}\|^2, \tag{102}$$

$$\varepsilon(c\bar{u}z(\theta_t\omega), A^\sigma\bar{v}) \leq \frac{\varepsilon|c||z(\theta_t\omega)|}{2\sqrt{\lambda_1}}(\|A^{\frac{1+\nu}{2}}\bar{u}\|^2 + \|A^{\frac{\nu}{2}}\bar{v}\|^2), \tag{103}$$

$$\alpha(c\nabla\bar{u}z(\theta_t\omega), \nabla A^\nu\bar{v}) \leq \frac{\alpha\lambda_1|c||z(\theta_t\omega)|}{2}\left(\|A^{\frac{1+\nu}{2}}\bar{u}\|^2 + \|A^{\frac{\nu}{2}}\bar{v}\|^2\right), \tag{104}$$

$$(c^2\bar{u}z^2(\theta_t\omega), A^\nu\bar{v}) \leq \frac{2|c|^4|z(\theta_t\omega)|^4}{\varepsilon\lambda_0} + \frac{\varepsilon}{8}(\|A^{\frac{1+\nu}{2}}\bar{u}\|^2 + \|A^{\frac{\nu}{2}}\bar{v}\|^2), \tag{105}$$

$$(c\bar{v}z(\theta_t\omega), A^\nu\bar{v}) \leq \frac{|c||z(\theta_t\omega)|}{2}\|A^{\frac{\nu}{2}}\bar{v}\|^2, \tag{106}$$

$$(c\bar{u}z(\theta_t\omega), A^\nu\zeta)_{\mu,1} \leq \frac{|c||z(\theta_t\omega)|}{2}(\|A^{\frac{1+\nu}{2}}\bar{u}\|^2 + \|A^{\frac{\nu}{2}}\zeta\|^2_{\mu,1}), \tag{107}$$

$$(g(x,t), A^\nu\bar{v}) \leq \frac{2}{(4\alpha + \varepsilon)}\|A^{\frac{\nu}{2}}g(x,t)\|^2 + \frac{4\alpha + \varepsilon}{8}\|A^{\frac{\nu}{2}}\bar{v}\|^2, \tag{108}$$

for nonlinear term we have

$$\begin{aligned} (f(u) - f_0(\tilde{u}), A^\nu\bar{v}) \quad = \quad & (f(u) - f_0(\bar{u}), A^\nu(\bar{u} + \varepsilon\bar{u} - c\bar{u}z(\theta_t\omega))) \\ \leq \quad & \frac{d}{dt}\int_{\mathbb{R}^n}(f(u) - f_0(\tilde{u}))A^\nu\bar{u}dx + \int_{\mathbb{R}^n}(f(u) - f_0(\tilde{u}))A^\nu\bar{u}dx \\ & - \int_{\mathbb{R}^n}(f'(u)u_t - f_0'(\tilde{u})\tilde{u}_t)A^\nu\bar{u}dx - \int_{\mathbb{R}^n}(f(u) - f_0(\tilde{u}))A^\nu\bar{u}z(\theta_t\omega)dx. \end{aligned}$$

Next, due to (8), (74)-(75), the Cauchy-Schwartz inequality and the Young inequality, we arrive at

$$\int_{\mathbb{R}^n}(f'(u)u_t - f_0'(\tilde{u})\tilde{u}_t)A^\nu \bar{u}dx = \int_{\mathbb{R}^n}((f_0'(u)-f_0'(\tilde{u}))u_t + f_0'(\tilde{u})\bar{u}_t + f_1'(u)u_t)A^\nu \bar{u}dx, \tag{109}$$

hence, the following inequalities holds

$$\begin{aligned}\int_{\mathbb{R}^n}(f_0'(u)-f_0'(\tilde{u}))u_tA^\nu\bar{u}dx &\leq C\int_{\mathbb{R}^\times} f_0''(u+\theta(u-\tilde{u}))|u-\tilde{u}||u_t||A^\nu\bar{u}|dx\\ &\leq C\int_{\mathbb{R}^n}\left(1+|u|^3+|\tilde{u}|^3\right)|\bar{u}||A^\nu\bar{u}||u_t|dx\\ &\leq C\left(1+\|u\|_{L^6}^3+\|\tilde{u}\|_{L^6}^3\right)\|\bar{u}\|_{L^{\frac{6}{1-2\nu}}}\|A^\nu\bar{u}\|_{L^{\frac{6}{1+2\nu}}}\|u_t\|_{L^6}\\ &\leq k_1(r,\tau-t,\omega)\|A^{\frac{1+\nu}{2}}\bar{u}\|\\ &\leq 4\varepsilon k_1^2(r,\tau-t,\theta_t\omega)+\frac{\varepsilon}{16}\|A^{\frac{1+\nu}{2}}\bar{u}\|^2,\end{aligned}\tag{110}$$

and note that $\nu\leq\frac{5-p}{2}$,

$$\begin{aligned}\int_{\mathbb{R}^n}f_1'(u)u_tA^\nu\bar{u}dx &\leq C\int_{\mathbb{R}^n}(1+|u|^p)|u_t||A^\nu\bar{u}|dx\\ &\leq C(1+\|u\|^p_{L^{\frac{6p}{5-2\nu}}})\|A^\nu\bar{u}\|_{L^{\frac{6}{1+2\nu}}}\|u_t\|_{L^2}\\ &\leq C(1+\|\nabla u\|^p)\|A^\nu\bar{u}\|_{L^{\frac{6}{1+2\nu}}}\|u_t\|_{L^6}\\ &\leq 4\varepsilon k_2^2(r,\tau-t,\theta_t\omega)+\frac{\varepsilon}{16}\|A^{\frac{1+\nu}{2}}\bar{u}\|^2,\end{aligned}\tag{111}$$

such as

$$\begin{aligned}\int_{\mathbb{R}^n}f_0'(\tilde{u})\bar{u}_tA^\nu\bar{u}dx &\leq C(1+\|\tilde{u}\|_{L^6}^4)\|A^{\frac{1+\nu}{2}}\bar{u}\|_{L^{\frac{6}{1+2\nu}}}\|A^\nu\bar{u}_t\|_{L^{\frac{6}{3+2\nu}}}\\ &\leq C(1+\|\tilde{u}\|_{L^6}^4)\|A^{\frac{1+\nu}{2}}\bar{u}\|_{L^{\frac{6}{1+2\nu}}}\|A^\nu\bar{u}_t\|_{L^{\frac{6}{3+2\nu}}}\\ &\leq 4\varepsilon k_3(r,\tau-t,\theta_t\omega)(\|A^{\frac{\nu}{2}}\bar{u}\|^2+|\varepsilon|^2)+\frac{\varepsilon}{16}\|A^{\frac{1+\nu}{2}}\bar{u}\|^2_{L^{\frac{6}{1+2\nu}}}\end{aligned}\tag{112}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n}(f(u)-f_0(\tilde{u}))|A^\nu\bar{u}||z(\theta_t\omega)|dx &\leq C\int_{\mathbb{R}^n}f'(u+\theta(u-\tilde{u}))|u-\tilde{u}||A^\nu\bar{u}||z(\theta_t\omega)|dx\\ &\leq C\int_{\mathbb{R}^n}\left(1+|u|^4+|\tilde{u}|^4\right)|\bar{u}||A^\nu\bar{u}||z(\theta_t\omega)|dx\\ &\leq C\left(1+\|u\|_{L^6}^4+\|\tilde{u}\|_{L^6}^4\right)\|\bar{u}\|_{L^{\frac{6}{1-2\nu}}}\|A^\nu\bar{u}\|_{L^{\frac{6}{1+2\nu}}}|z(\theta_t\omega)|\\ &\leq 4\varepsilon\left(k_4^2(r,\tau-t,\theta_t\omega)+|z(\theta_t\omega)|^2\right)+\frac{\varepsilon}{16}\left\|A^{\frac{1+\nu}{2}}\bar{u}\right\|^2.\end{aligned}\tag{113}$$

Thus, by collecting all (100)-(113) and (99), we show that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|A^{\frac{\nu}{2}}\varphi_2\|_E^2+2\left(f(u)-f_0(\tilde{u})\right)\right)+\frac{\varepsilon}{4}\left\|A^{\frac{\nu}{2}}\varphi_2\right\|_E^2+\frac{k\varepsilon}{2}(f(u)-f_0(\tilde{u}))\\ &\leq \mu_1|c||z(\theta_t\omega)|\|A^{\frac{\nu}{2}}\varphi_2\|_E^2+C(\omega)[1+k_1^2(r,\tau-t,\omega)+k_2^2(r,\tau-t,\theta_t\omega)\\ &+k_3^2(r,\tau-t,\omega)+k_4^2(r,\tau-t,\omega)+|z(\theta_t\omega)|^2+|z(\theta_t\omega)|^4+\|A^{\frac{\nu}{2}}q(x,t)\|^2].\end{aligned}\tag{114}$$

By Gronwall's inequality to (114) on $[\tau-t,r]$ and replacing ω to $\theta_{-\tau}\omega$, we have

$$\begin{aligned}\left\|A^{\frac{\nu}{2}}\varphi_2(r,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\right\|_E^2 \leq &\left(\|A^{\frac{\nu}{2}}\varphi_2(r,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t})\|_E^2+2(f(u(r,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}))\right.\\ &-f_0(\tilde{u}(r,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}))\end{aligned}$$

$$\leq \left(\|A^{\frac{\nu}{2}}\varphi_2\|_E^2 + (f(u) - f_0(\tilde{u}))\right) exp^{2\int_r^{\tau-t}(\sigma-\mu_1|c||z(\theta_{s-\tau}\omega)|))(s-\tau,\omega)ds}$$
$$+ \int_{\tau-t}^{r} \rho_1(s,\theta_s\omega) exp^{2\int_r^s(\sigma-\mu_1|c||z(\theta_{\varsigma-\tau}\omega)|)(\varsigma-\tau,\omega)d\varsigma} ds. \quad (115)$$

We put

$$\rho_1(r,\theta_t\omega) = C(\omega)[1 + k_1^2(r,\tau-t,\theta_t\omega) + k_2^2(r,\tau-t,\theta_t\omega) + k_3^2(r,\tau-t,\theta_t\omega)$$
$$+k_4^2(r,\tau-t,\theta_t\omega) + |z(\theta_t\omega)|^2 + |z(\theta_t\omega)|^4 + \|A^{\frac{\nu}{2}}g(x,t)\|^2]. \quad (116)$$

Similar to above equation,

$$\begin{aligned}
\int_{\mathbb{R}^n} (f(u) - f_0(\tilde{u}))\, A^{\nu}\bar{u}dx &\leq C\int_{\mathbb{R}^n} (f'(u+\theta(u-\tilde{u}))|u-\tilde{u}||A^{\nu}\bar{u}|dx \\
&\leq C\int_{\mathbb{R}^n} \left(1 + |u|^4 + |\tilde{u}|^4\right)|\bar{u}||A^{\nu}\bar{u}|dx \\
&\leq C(1 + \|u\|_{L^6}^4 + \|\tilde{u}\|_{L^6}^4)\|\bar{u}\|_{L^{\frac{6}{1-2\nu}}}\|A^{\nu}\bar{u}\|_{L^{\frac{6}{1+2\nu}}} \\
&\leq k_5(\tau,\tau-t,\theta_{-t}\omega)\,\|A^{\frac{1+\nu}{2}}\bar{u}\|\|A^{\nu}\bar{u}\| \\
&\leq \varepsilon k_5^2(\tau,\tau-t,\theta_{-t}\omega)\,\|A^{\frac{\nu}{2}}\bar{u}\|^2 + \frac{\varepsilon}{4}\|A^{\frac{1+\nu}{2}}\bar{u}\|^2, \quad (117)
\end{aligned}$$

by (115) and (117), we can get

$$\|A^{\nu}\varphi_2(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t})\|_E^2 \leq \varrho_2^2(\tau,\omega).$$

□

6. Random attractors

In this section, we establish the existence of a $\mathcal{D}$-random attractor for the random dynamical system Φ associated with system (18) on $\mathbb{R}^n$, that is, by Lemma 10, Φ has a closed random absorbing set in $\mathcal{D}$, which along with the $\mathcal{D}$-pullback asymptotic compactness, they imply the existence of a unique $\mathcal{D}$-random attractor. Next due to decomposition of solutions we shall prove the $\mathcal{D}$-pullback asymptotic compactness of Φ (see[10,37]). For $\tau \in \mathbb{R}, \omega \in \Omega, t \geq 0$, we get

$$\zeta(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t},s) = \begin{cases} \bar{u}((\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t} - \bar{u}(\tau-r,\tau-t,\theta_{-\tau+s}\omega,\varphi_{\tau-t}),\ r \leq t, \\ \bar{u}(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}),\ t \leq r; \end{cases} \quad (118)$$

$$\zeta_s(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}) = \begin{cases} \bar{u}_t(\tau-r,\tau-t,\theta_{-\tau+r}\omega,\varphi_{\tau-t}),\ 0 \leq r \leq t, \\ 0,\quad t \leq r. \end{cases} \quad (119)$$

Lemma 16. *Let $E_\nu = H_{2\nu+1} \times H_{2\nu} \times L_\mu^2(\mathbb{R}^+, H_{2\nu+1}) \to L_\mu^2(\mathbb{R}^+, H_{2\nu+1})$ is projection operator setting, $Y = \psi(r, B_\nu(\tau,\omega))$ is a random bounded absorbing set, by Lemma 15, $\psi(r)$ is the solution of the system (76), and by Lemma 15, there is a positive random radius $\varrho_\nu(\tau,\omega)$ depending on r, such that*

$$\begin{cases} 1\ \ Y \text{ is bounded in } L_\mu^2(\mathbb{R}^+, H_{1+2\nu}) \bigcap H_\mu^1(\mathbb{R}^+, H_{2\nu}), \\ 2\ \ \sup\limits_{\eta\in Y, s\in\mathbb{R}^+} \|\eta(s)\|_{\mu,1}^2 \leq \varrho_\nu(\tau,\omega). \end{cases} \quad (120)$$

Denote by B_ν the closed ball of $H_{1+2\nu} \times H_{2\nu}$ of random variable radius $\varrho_\nu(\tau,\omega)$, let we apply on a finite domain. B_ν is compact subset of $H_{1+2\nu} \times H_{2\nu}$. Thus, we chose that a set $\tilde{B}_\nu(\tau,\omega)$

$$\tilde{B}_\nu(\tau\,,\omega) = \overline{\bigcup_{\psi_{\tau-t}(\theta_{-\tau}\omega)\in B_1(\theta_{-t}\omega)}\bigcup_{t\geq 0}\zeta(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega),s)},\ s \in \mathbb{R}^+,\ \tau \in \mathbb{R},\ \omega \in \Omega, \quad (121)$$

hence, v is as in (97). From (3) and (120), we find

$$\|\eta(s)\|_{\mu,1}^2 = \int_0^{+\infty} \mu(s)\|\nabla\eta(s)\|^2 ds \leq \varrho_v(\tau,\omega)\int_0^{+\infty} e^{\delta s}ds \leq \frac{\varrho_v(\tau,\omega)}{\delta}. \tag{122}$$

The next Lemma we investigate the main result about the existence of a random attractor for random dynamical system Φ.

Lemma 17. *we assume that $\psi(t,\tau,\omega)$ is a solution of system (77) and the conditions of Lemma 14 hold, for each $t \geq 0$, there exists a random set $\tilde{B}_v(\omega) \in \mathcal{D}(E_v)$ with*

$$\|\tilde{B}_v(\tau,\omega)\|_{E_v} = \sup_{\tilde{\psi}\in\tilde{B}_v(\tau,\omega)} \|\tilde{\psi}\|_E \leq \tilde{M}(\tau,\omega)$$

is relatively compact in E. Then we show the following attraction property of $\tilde{A}(\tau,\omega)$, for every $B(\tau,\theta_{-t}\omega) \in \mathcal{D}(E)$, if there exist and positive number σ and $\tilde{M}(\tau,\omega) \geq 0$ so as for each $\tau \in \mathbb{R}, \omega \in \Omega$ it satisfy

$$d_H(\Phi(t,\tau-t,\theta_{-t}\omega, B_1(\tau-t,\theta_{-t}\omega)), \tilde{B}_1(\tau,\omega)) \leq M_0(\tau,\omega)e^{-\sigma t} \rightarrow 0 \text{ at } t \rightarrow +\infty \tag{123}$$

Proof. Let $\varphi_{\tau-t}(\theta_{-\tau}\omega) \in B_1(\tau-t,\theta_{-t}\omega)$ and by (118)-(120) and Lemma 17, it concludes that $\tilde{B}_v(\tau\ ,\omega)$ is relatively compact in $L_\mu^2(\mathbb{R}^+, H_1)$, let $B_v(\omega) \subset E_v \subset E$ be the ball of E_v of radius $M(\tau,\omega)$ defined by (27), where v is as in (97). Lastly, we get compact set $A_0(\varsigma,w) = \tilde{B}_v \times B_v \subset E$.
Since Lemma 10, Lemma 14 and $\varphi_{\tau-t}(\theta_{-\tau}\omega) \in B_0(\tau-t,\theta_{-t}\omega)$, there exists a random set $M(\tau,\omega) \in B_0 \subseteq B(\tau,\omega) \in \mathcal{D}(E)$, such that

$$d_H(\Phi(t,\tau-t,\theta_{-t}\omega, B(\tau-t,\theta_{-t}\omega)), B_0(\omega)) \leq M(\tau,\omega)e^{-\sigma t} \rightarrow 0 \text{ at } t \rightarrow +\infty, \tag{124}$$

next, follows from Lemma 13, for $\varphi_{\tau-t}(\theta_{-\tau}\omega) \in B_1(\tau-t,\theta_{-t}\omega)$, there exists positive a random variable $M_0(\tau,\omega) \in B_1(\tau,\omega) \in \mathcal{D}(E)$ and $M_1(\tau,\omega) \in B_1(\omega) \in \mathcal{D}(E)$ such that,

$$d_H(\Phi(t,\tau-t,\theta_{-t}\omega, B(\tau-t,\theta_{-t}\omega)), B_1(\tau,\omega)) \leq M_0(\tau,\omega)e^{-\sigma_1 t} \rightarrow 0 \text{ as } t \rightarrow +\infty, \tag{125}$$

by Lemma 15, let $\varphi_{\tau-t}(\theta_{-\tau}\omega) \in B_1(\tau-t,\theta_{-t}\omega)$, there exists positive a random variable $\varrho_2^2(\tau,\omega) \in B_1(\omega) \in \mathcal{D}(E)$, such that

$$d_H(\Phi(t,\tau-t,\theta_{-t}\omega, B(\tau-t,\theta_{-t}\omega)), B_1(\tau,\omega)) \leq \varrho_2^2(\tau,\omega)e^{-\sigma_1 t} \rightarrow 0 \text{ as } t \rightarrow +\infty, \tag{126}$$

let $v \geq 0$ is fixed, by above recursion of finite steps at most $\frac{1}{v}+2$, there exists random set $\tilde{\varrho}_v \in \tilde{B}_v(\omega) \in \mathcal{D}(E_v)$ as for as

$$d_H(\Phi(t,\tau-t,\theta_{-t}\omega, B_1(\tau-t,\theta_{-t}\omega)), \tilde{B}_v(\tau,\omega)) \leq \tilde{\varrho}_v(\tau,\omega)e^{-\sigma_{2v} t} \rightarrow 0 \text{ at } t \rightarrow +\infty, \tag{127}$$

due to Lemma 16 and (118)-(120), $\varrho_v(\tau,\omega) \in B_1(\omega) \in \mathcal{D}(E)$ we have

$$d_H(\Phi(t,\tau-t,\theta_{-t}\omega, B_v(\tau-t,\theta_{-t}\omega)), \tilde{B}_v(\tau,\omega)) \leq \varrho_v(\tau,\omega)e^{-\sigma_{2v} t} \rightarrow 0 \text{ at } t \rightarrow +\infty, \tag{128}$$

and

$$\tilde{A}(\tau,\omega) = B_v(\tau,\omega) \times \tilde{B}_v(\tau,\omega), \tag{129}$$

Thus, by Lemma 11, there exists $T = T(\tau,\omega,B) \geq 0$ such that $\varphi(t,\tau-t,\theta_{-\tau}\omega, B(\tau-t,\theta_{-\tau}\omega) \subseteq B_0(\omega))\ \forall t \geq T$
Let $t \geq T$ and $T = t-r \geq T(\tau,\omega,B_0) \geq 0$, using cocycle property (3) of Φ, we show that

$$\begin{aligned} \varphi(t,\tau-t,\theta_{-\tau}\omega, B(\tau-t,\theta_{-\tau}\omega)) &= \varphi(t,\tau-T,\theta_{-\tau}\omega, B(\tau-T,\theta_{-\tau}\omega)) \\ &= \varphi(\tau,\tau-\hat{T},(\theta_{-\tau}\omega),\varphi(\tau-\hat{T},\tau-T,\theta_{-\tau}\omega, B(\tau-T,\theta_{-\tau}\omega)) \\ &\subseteq \varphi(\tau-T,\tau-T,\theta_{-\tau}\omega, B_0(\theta_{-T}\omega)) \subseteq B_1(\tau,\omega), \end{aligned} \tag{130}$$

for each $\varphi(\tau,\tau-t,(\theta_{-\tau}\omega),\varphi_{\tau-t}(\theta_{-\tau}\omega)) \in \varphi(t,\tau-t,\theta_{-\tau}\omega,B(\tau-t,\theta_{-t}\omega))$, for $t \geq r+T(\tau,\omega,B_0)$, where $\varphi_{\tau-t}(\theta_{-\tau}\omega) \in B(\tau-t,\theta_{-t}\omega)$. By (97) and Lemma 15, we get

$$\begin{aligned}\tilde{\varphi}(\tau,\tau-t,(\theta_{-\tau}\omega),\varphi_{\tau-t}(\theta_{-\tau}\omega)) &= \varphi(\tau,\tau-t,(\theta_{-\tau}\omega),\varphi_{\tau-t}(\theta_{-\tau}\omega)) \\ &\quad -\psi(\tau,\tau-t,(\theta_{-\tau}\omega),\psi_{\tau-t}(\theta_{-\tau}\omega)) \in \tilde{A}(\tau,\omega). \end{aligned} \tag{131}$$

Therefore, thanks to Lemma 14, we conclude that

$$\begin{aligned}\inf_{\tilde{\psi}\in\tilde{A}(\tau,\omega)} \|\varphi(\tau,\tau-t,(\theta_{-\tau}\omega),\varphi_{\tau-t}(\theta_{-\tau}\omega))-\tilde{\psi}\|_E^2 &\leq \|\psi(\tau,\tau-t,(\theta_{-\tau}\omega),\psi_{\tau-t}(\theta_{-\tau}\omega))\|_E^2 \\ &\leq \tilde{M}^2(\tau,\omega)e^{-\sigma t}\ ,\ \forall t > \tilde{T}+T(\tau,\omega,B_0),\end{aligned}$$

so

$$d_H(\Phi(t,\tau-t,\theta_{-t}\omega,B(\tau-t,\theta_{-\tau}\omega)),\tilde{A}(\tau,\omega)) \leq \tilde{M}(\tau,\omega)e^{-\sigma_1 t} \ \to 0 \ at\ t\to+\infty.$$

□

Theorem 18. *Suppose that (6)-(11) hold. Then the continuous cocycle Φ associated with the problem (15)-(17) or (18) has a unique $\mathcal{D}$-pullback attractor $\mathcal{A} \subseteq \tilde{A}(\tau,\omega)\bigcap B_0(\omega)$, $\mathcal{A}=\{\mathcal{A}(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$ in $\mathbb{R}^n$.*

Proof. Hence that the continuous cocycle Φ has a closed random absorbing set $\{A(\omega)\}_{\omega\in\Omega}$ in $\mathcal{D}$, by Lemma 10, Lemma 11 and Lemma 16, the continuous cocycle Φ is $\mathcal{D}$-pullback asymptotically compact in $\mathbb{R}^n$. Since that the existence of a unique $\mathcal{D}$- random attractor for Φ follows from Lemma 8 immediately. □

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