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Reconciling Power-law Slopes in Solar Flare and Nanoflare Size Distributions

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Abstract

We unify the power laws of size distributions of solar flare and nanoflare energies. We present three models that predict the power-law slopes α_E of flare energies defined in terms of the 2D and 3D fractal dimensions (D_A, D_V): (i) the spatiotemporal standard self-organized criticality model, defined by the power-law slope $\alpha_{E1} = 1 + 2/(D_V + 2) = (13/9) \approx 1.44$; (ii) the 2D thermal energy model, $\alpha_{E2} = 1 + 2/D_A = (7/3) \approx 2.33$; and (iii) the 3D thermal energy model, $\alpha_{E3} = 1 + 2/D_V = (9/5) \approx 1.80$. The theoretical predictions of energies are consistent with the observational values of these three groups, i.e., $\alpha_{E1} = 1.47 \pm 0.07$, $\alpha_{E2} = 2.38 \pm 0.09$, and $\alpha_{E3} = 1.80 \pm 0.18$. These results corroborate that the energy of nanoflares does not diverge at small energies, since ($\alpha_{E1} < 2$) and ($\alpha_{E3} < 2$), except for the 2D model ($\alpha_{E2} > 2$). Thus, while this conclusion does not support nanoflare scenarios of coronal heating from a dimensionality point of view, magnetic reconnection processes with quasi-1D or quasi-2D current sheets cannot be ruled out.

Unified Astronomy Thesaurus concepts: Active solar corona (1988); Solar flares (1496); Solar x-ray flares (1816)

1. Introduction

The concept of self-organized criticality (SOC) specifies nonlinear (avalanching) phenomena based on next-neighbor interactions in a lattice grid (Bak et al. 1987, 1988). The spatiotemporal evolution of avalanches in such complex environments has been numerically simulated by cellular automaton methods, which exhibit fractal structures (Pruessner 2012). Alternatively, SOC-related avalanches can be considered analytically, as instabilities with a nonlinear initial growth phase and subsequent saturation (Rosner & Vaiana 1978; Aschwanden 2011, 2012, 2014).

Let us introduce the first definition of an SOC energy, which we call the spatiotemporal energy of an SOC avalanche, labeled as E_1 in this Letter. The analytical SOC approach takes both the spatial as well as the temporal evolution of an SOC avalanche into account. Hence, the total energy of an SOC avalanche is determined from the spatial and temporal integration over the unstable 2D areas (or 3D volumes) in the paradigm of sand pile avalanches (Bak et al. 1987, 1988). Theoretical predictions of SOC parameters based on the size distribution of avalanche energies are described in Section 2 and are summarized in Table 1. The necessary parameters to characterize the spatiotemporal energy requires the measurement of the mean flux F and the time duration T of an event, while the spatiotemporal energy is defined by $E_1 = F \times T$. Measurements of spatiotemporal energies were originally applied to soft X-rays in solar flares (Drake 1971), hard X-rays (Crosby et al. 1993; Lu et al. 1993), gamma rays (Perez Enriquez & Miroshnichenko 1999), as well as to EUV small-scale brightenings or nanoflares (Brkovic et al. 2001; Uritsky et al. 2013). We compile these data sets in Table 2.

The second definition of an SOC energy was introduced by the 2D definition of the thermal energy in a high-temperature plasma (called E_2 here), i.e., $E_2 = 3 k_B T_e n_e V$ (where k_B is the Boltzmann constant, T_e is the electron temperature, n_e is the electron density, and V is the fractal volume). The avalanche volumes of solar flares and nanoflares are generally close to fractal geometries, if they are measured on a pixel-by-pixel basis (Aschwanden & Aschwanden 2008a, 2008b), in contrast to an encompassing (nonfractal) circle or square (with length scale L). An additional relationship is the definition of the emission measure, $EM = n_e^2 V$, which can be used to substitute the electron density, i.e., $n_e = \sqrt{EM/V}$. Furthermore, a relationship for the (fractal) event volume V needs to be specified. The projected area A, which is fractal, can be measured directly in the image plane, but the line-of-sight depth is unknown, which led some pioneers to quantify it with a constant height h_0, leading to the expression $V = A\, h_0$ for the volume (Krucker & Benz 1998; Parnell & Jupp 2000; Benz & Krucker 2002; Table 3). In hindsight, this choice of a constant depth h_0 has been criticized because it is very unlikely that the line-of-sight depth is equal from the smallest nanoflares (of the size of a solar granulation cell) to the largest flares (of the size of an active region). Furthermore, the assumption of a constant height h_0 introduces a crucial bias that modifies the power-law slope of the energy size distribution substantially, from $\alpha_{E3} = 1.80$ to $\alpha_{E2} = 2.33$, across the critical value of $\alpha_E = 2$, as we will see in the remainder of this Letter.

A third definition of an SOC energy (E_3) was made by abandoning the assumption of a constant height, i.e., $V = A\, h_0$, and instead replacing it with a more physical assumption of an isotropic volume, i.e., $V = A^{3/2}$, which corresponds to a line-of-sight depth of $h = \sqrt{A}$, while D_A is the fractal dimension of the area, $A = L^{D_A}$. This approach, which we call the 3D thermal energy model (Table 4), has been applied frequently (Shimizu 1995; Berghmans et al. 1998; Berghmans & Clette 1999; Parnell & Jupp 2000; Aschwanden & Parnell 2002; Uritsky et al. 2007; Joulin et al. 2016; Nhalil et al. 2020; Kawai & Imada 2022; Purkhart & Veronig 2022).

In this Letter we calculate the power-law indices α_E of the energy size distributions in a unified way, which provides us a diagnostic whether flare and nanoflare energies diverge at the lower or upper end of the size distribution, and this way we can

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Table 1
Parameters of the Standard SOC Model, with Fractal Dimensions D_x and Power-law Slopes α_x of Size Distributions)

Parameter	Power-law Slope Analytical	Power-law Slope Numerical
Euclidean dimension	$d =$	3.00
Diffusion type	$\beta =$	1.00
Area fractal dimension	$D_A = d - (3/2) =$	1.50 = (3/2)
Volume fractal dimension	$D_V = d - (1/2) =$	2.50 = (5/2)
Length	$\alpha_L = d =$	3.00
Area	$\alpha_A = 1 + (d - 1)/D_A =$	2.33 = (7/3)
Volume	$\alpha_V = 1 + (d - 1)/D_V =$	1.80 = (9/5)
Duration	$\alpha_T = 1 + (d - 1)\beta/2 =$	2.00
Mean flux	$\alpha_F = 1 + (d - 1)/(\gamma D_V) =$	1.80 = (9/5)
Peak flux	$\alpha_P = 1 + (d - 1)/(\gamma d) =$	1.67 = (5/3)
Spatiotemporal energy	$\alpha_{E_1} = 1 + (d - 1)/(\gamma D_V + 2/\beta) =$	1.44 = (13/9)
Thermal energy (h = const)	$\alpha_{E_2} = 1 + 2/D_A =$	2.33 = (7/3)
Thermal energy (h = A $^{1/2}$)	$\alpha_{E_3} = 1 + 2/D_V =$	1.80 = (9/5)

assess the importance (or nonimportance) of coronal heating by nanoflares. The mathematical derivation of the SOC models is given in Section 2, a discussion in Section 3, and conclusions in Section 4.

2. Theoretical Models and Observations

An SOC model should be able to predict the (occurrence frequency) size distribution functions, which can be formulated in terms of power-law function slopes α_x to first order. Common SOC parameters $x = [L, A, V, T, F, P, E]$ include the length scale L, the 2D area A, the 3D volume V, the time duration T, the mean flux or intensity F, the peak flux or peak intensity P, and the fluence or energy E. In this study we reconcile three different definitions of the energy (E) that have been used in the past, namely, the spatiotemporal definition of the standard SOC model (E_1), the 2D fractal model (E_2), and the 3D fractal model (E_3). A glossary of SOC terms are listed in the Appendix.

2.1. The Standard SOC Model

The standard SOC model is derived from first principles in previous studies (Aschwanden 2012, 2014, 2022). A brief summary of the calculations that clarify the assumptions made here is given in the following, while a more detailed description is provided in Aschwanden (2022).

We start with the size distribution $N(L)$ of length scales L, also called the scale-free probability conjecture,

$$N(L)\, dL \propto L^{-d}\, dL, \tag{1}$$

where d is the Euclidean space dimension, which is set to $d = 3$ for most real-world data. The power-law indices α_x of the size distribution functions can then be calculated for every SOC parameter x by variable substitution $L \mapsto x$,

$$N(x)\, dx = L(x)^{-d} \left(\frac{dL}{dx}\right) dx = x^{-\alpha_x}\, dx, \tag{2}$$

which just requires the scaling law $x(L)$ as a function of the length scale L, the inverted scaling law function $L(x)$, and its derivative dL/dx. Thus, we need to make an assumption of a scaling law $x(L)$ of each SOC parameter of interest.

For the spatial parameters we define the fractal dimensions for 2D areas A, which is identical to the fractal Hausdorff dimension $D_A \approx 3/2$,

$$A = L^{D_A}, \tag{3}$$

and likewise for the 3D volume V, which is identical to the fractal Hausdorff dimension $D_V \approx 5/2$,

$$V = L^{D_V}. \tag{4}$$

We can estimate the numerical values of the fractal dimensions D_A and D_V from the mean of the minimum and maximum values in each Euclidean domain,

$$D_A = \frac{(D_{A,\min} + D_{A,\max})}{2} = \frac{(1 + 2)}{2} = \frac{3}{2} = 1.50, \tag{5}$$

and correspondingly,

$$D_V = \frac{(D_{V,\min} + D_{V,\max})}{2} = \frac{(2 + 3)}{2} = \frac{5}{2} = 2.50. \tag{6}$$

In the standard SOC model we need four more scaling laws. The time duration T of an SOC avalanche can be linked to spatial (fractal) structures by the diffusive behavior,

$$T \propto L^{2/\beta}, \tag{7}$$

where the coefficient is $\beta = 1$ for classical diffusion, $\beta < 1$ for subdiffusive transport, and $\beta > 1$ for hyperdiffusive transport (also called Lévy flight). Furthermore, we need a relationship between the mean flux F and the emitting volume V,

$$F \propto V^{\gamma} = L^{D_V \gamma}, \tag{8}$$

which is generally found to be near to proportional; hence, we set $\gamma = 1$. A relationship between the peak flux P and the length scale L is

$$P \propto V^{\gamma} = L^{d\gamma}, \tag{9}$$

where the flux F (Equation (8)) is maximized to the peak flux P, i.e., $P(t_{peak}) = max[F(t)]$, by replacing the dimension D_V in Equation (8) with the Euclidean dimension d, i.e., $D_V \mapsto d$.

Finally, the fluence or energy E_1, which is expressed by the product of the mean flux F and the event duration T (for a spatiotemporal SOC event) yields (Crosby et al. 1993)

$$E_1 = (F \times T) \propto L^{(D_V \gamma + 2/\beta)}. \tag{10}$$

If we assume classical diffusion ($\beta = 1$) and flux-volume proportionality ($\gamma = 1$), the four basic scaling laws are reduced further to $T \propto L^2$ (Equation (7)), $F \propto L^{2.5}$ (Equation (8)), $P \propto L^3$ (Equation (9)), and $E_1 \propto L^{4.5}$ (Equation (10)). In this framework, there are no free parameters, and the power-law slopes α_x of the size distributions,

$$N(x)\, dx = x^{-\alpha_x}\, dx, \tag{11}$$

Table 2
Observed Frequency Distributions of Spatiotemporal Energies $E = F * T$, by Integrating the Flux Rate F in Space and Time T

Power-law Slope of Energy α_E	Instrument Wavelength	Observed Phenomenon	Reference
1.53 ± 0.02	HXRBS(>25 keV)	solar flares	Crosby et al. (1993)
1.51 ± 0.04	HXRBS(>25 keV)	solar flares	Crosby et al. (1993)
1.48 ± 0.02	HXRBS(>25 keV)	solar flares	Crosby et al. (1993)
1.53 ± 0.02	HXRBS(>25 keV)	solar flares	Crosby et al. (1993)
1.51	ISEE3(>25 keV)	solar flares	Lu et al. (1993)
1.39 ± 0.01	PHEBUS(>100 keV)	solar flares	Perez Enriquez & Miroshnichenko (1999)
1.44	Explorer SXR 2-12 A	solar flares	Drake (1971)
1.34 ± 0.08	SMM/FCS, OV	blinkers	Brkovic et al. (2001)
1.50 ± 0.04	SOHO/EIT 195,HMI	EUVE events	Uritsky et al. (2013)
1.47 ± 0.07[a]			Mean of nine observations
1.44			Theoretical prediction

Note.
[a] Bold font indicates theoretical values, while roman font indicates observational values.

of all SOC parameters $x = [A, V, T, F, P, E]$ can be predicted by variable substitution (Equation (2)), yielding the values $D_A = 3/2$, $D_V = 5/2$, $\alpha_A = 7/3 \approx 2.33$, $\alpha_V = 9/5 \approx 1.80$, $\alpha_T = 2$, $\alpha_F = 9/5 \approx 1.80$, $\alpha_P = 5/3 \approx 1.67$, and $\alpha_{E_1} = 13/9 \approx 1.44$, as listed in Table 1.

Comparison of these theoretical predictions of power-law slopes $\alpha_x^{\rm theo}$ with observed size distributions $\alpha_x^{\rm obs}$ have been presented in Aschwanden (2022). Among the solar flare data sets that apply the spatiotemporal energy model (Equation (10); Table 2), we identify hard X-ray data (Crosby et al. 1993; Lu et al. 1993), gamma-ray data (Perez Enriquez & Miroshnichenko 1999), soft X-ray data (Drake 1971), and EUV data (Brkovic et al. 2001; Uritsky et al. 2013), which exhibit a mean power slope of $\alpha_{E_1}^{\rm obs} = 1.47 \pm 0.07$, agreeing well with the theoretical prediction $\alpha_{E_1}^{\rm theo} = (13/9) \approx 1.44$ (Table 2).

2.2. The 2D Thermal Energy Model

The energy of a spatiotemporal SOC event is defined in the standard SOC model by the product of the count rate (F) and the event duration (T) (Equation (10)), which is appropriate for nonthermal energies that are quantified by hard X-ray counts (or intensity) in solar and stellar flares. In both solar or stellar flares, down to nanoflares, one can estimate thermal (radiative) energies at the peak time of an event, defined by

$$E_2 = (3 k_B n_e T_e)\ V, \tag{12}$$

where k_B is the Boltzmann constant, n_e is the electron density, T_e is the electron temperature, and V is the 3D volume, all measured at the peak time of an event. The 3D volume V cannot be measured directly, which led some authors to approximate the volume with a constant height h_0 in the line of sight,

$$V = A\ h_0 = L^{D_A}\ h_0, \tag{13}$$

while the fractal area is defined as $A = L^{D_A}$. The fractality is not explicitly mentioned in some of these studies, but every pattern recognition code that measures an area on a pixel-by-pixel basis (at different spatial resolutions) yields approximately the fractal area $A \propto L^{D_A}$ with $D_A < 2$, rather than the encompassing Euclidean area $A = L^2$. Inserting the area fractal dimension $D_A = 3/2$ (Equation (5)) into the expression for the thermal energy $E_2 \propto L^{D_A}$ (Equation (12)), we obtain

$$E_2 = (3 k_B n_e T_e h_0)\ L^{D_A}. \tag{14}$$

The same way as we substituted the variable L in the size distribution with the energy $x = E$ (Equations (1) and (2)),

$$N(E_2)\ dE_2 = L(E_2)^{-d} \left(\frac{dL}{dE_2}\right) dE_2 = E_2^{-\alpha_{E_2}}\ dE_2, \tag{15}$$

yielding the power-law slope α_{E_2}, for $d = 3$ and $D_A = 3/2$,

$$\alpha_{E_2} = 1 + \frac{(d-1)}{D_A} = \frac{7}{3} \approx 2.33. \tag{16}$$

Note that we treat the variables n_e, T_e, and h_0 as constants here, while the scaling law hinges entirely on the correlation between the thermal energy E_2 and the length scale L, rendering a first-order approximation to the power-law slope α_{E_2}. Since the thermal energy $E_2 \propto V \propto L^{D_A}$ (Equation (14)) and the fractal area $A \propto L^{D_A}$ (Equation (13)) have the same scaling law, the power-law index for the size distribution of areas α_A has the same power-law index α_{E_2} too,

$$\alpha_A = \alpha_{E_2} = \frac{7}{3} \approx 2.33. \tag{17}$$

The definition of the energy made here (Equation (12)) invokes an isothermal plasma. Nevertheless, the definition of the thermal energy can accommodate a multithermal formalism, which involves a differential emission measure distribution function $dEM(T_e)/dT_e$, characterized by the increase in the emission measure EM, the (mean) electron density n_e, and the volume V,

$$EM = n_e^2\ V, \tag{18}$$

which inserted into Equation (12) yields

$$E_2 = (3 k_B T_e)\ \sqrt{EM \times V} = (3 k_B T_e)\ \sqrt{EM \times A\ h_0}. \tag{19}$$

Size distribution of thermal energies, based on emission measure changes EM, yields power-law slopes of $\alpha_{E_2} = 2.38 \pm 0.09$ (Krucker & Benz 1998; Parnell & Jupp 2000; Benz & Krucker 2002), which match closely the theoretically expected value of $\alpha_E = 2.33$ (Equation (16); Table 3).

Table 3
Observed Frequency Distributions of Thermal Energies E_2 Calculated from Peak Emission Measures and Temperatures with 2D Fractal Model and Constant Line-of-sight Depth ($h_0 = \text{const}$)

Power-law Slope of Fluence or Energy α_{E_2}	Instrument Wavelength	Observed Phenomenon	Reference
2.45 ± 0.15	EIT 171,195	EUV transient	Krucker & Benz (1998)
2.30 ± 0.30	TRACE 171,195	Nanoflares	Parnell & Jupp (2000)
2.48 ± 0.11	TRACE 171,195	Nanoflares	Parnell & Jupp (2000)
2.31	EIT 171,195	EUV transient	Benz & Krucker (2002)
2.38 ± 0.09[a]			Mean of four observations
2.33			Theoretical prediction

Note.
[a] Bold font indicates theoretical values, while roman font indicates observational values.

2.3. The 3D Thermal Energy Model

In the 3D version of the thermal model, the SOC avalanche volume $V \propto L^{D_V}$ (Equation (4)) is defined by the (mean) Hausdorff dimension $D_V = (5/2)$ (Equation (6)), which inserted into the expression for the thermal energy is

$$E_3 = (3 k_B n_e T_e)\, L^{D_V}. \tag{20}$$

We substitute the variable L in the size distribution of the thermal energy E_3 (Equation (14)),

$$N(E_3)\, dE_3 = L(E_3)^{-d}\, \frac{dL}{dE_3}\, dE_3 = E_3^{-\alpha_{E_3}}\, dE_3, \tag{21}$$

yielding the power-law slope α_{E_3}, for $d = 3$ and $D_V = 5/2$,

$$\alpha_{E_3} = 1 + \frac{(d-1)}{D_V} = \frac{9}{5} = 1.80. \tag{22}$$

Note that the power-law slope is substantially steeper in the 2D model ($\alpha_{E_2} = 2.33$) than in the 3D version ($\alpha_{E_3} = 1.80$). Moreover, the two models predict power-law slopes below ($\alpha_{E_2} < 2$), as well as above ($\alpha_{E_3} > 2$) the critical value of $\alpha_E = 2$, which decides whether the nanoflare population diverges at the low end or upper end of the size distribution. Calculations of the multithermal energy using a 3D model have been performed using Yohkoh, TRACE, SOHO, AIA, and IRIS data (Shimizu 1995; Berghmans et al. 1998; Berghmans & Clette 1999; Parnell & Jupp 2000; Aschwanden & Parnell 2002; Uritsky et al. 2007; Joulin et al. 2016; Nhalil et al. 2020; Kawai & Imada 2022; Purkhart & Veronig 2022), as listed in Table 4.

3. Discussion

3.1. Scaling Laws

Scaling laws, typically expressed by variables (x, y, …) with power-law dependencies, $x^\alpha y^\beta \ldots = \text{const}$, are powerful tools to test parameter correlations and size distribution functions. If a scaling law function $y(x)$ and a single size distribution $N(x)$ are known, we can derive the size distribution $N(y)$ of a correlated parameter by variable substitution, $N(y)dy = N(x[y])(dx/dy)dy$. This way we can predict theoretical size distributions $N(y)$ based on observed size distributions $N(x)$, as well as significant correlations between variables. Here we explore the size distributions of nine variables $x = [L, A, V, T, F, P, E_1, E_2, E_3]$ in a unified scheme (Table 1). We focus mainly on the three energy parameters $x = [E_1, E_2, E_3]$, which represent the spatiotemporal energy (E_1), and the 2D (E_2) and 3D fractal thermal energies (E_3). Additional forms of energy definitions in solar and stellar flares (such as magnetic energies, radiative energies, conductive energies, coronal mass ejection kinetic or potential energies, etc.) are studied elsewhere (e.g., Aschwanden et al. 2017). The fact that we can predict energy size distribution functions, $[N_{E_1}, N_{E_2}, N_{E_3}]$, within the statistical uncertainties, corroborates the validity of the unified scaling laws derived here. Specifically, the scaling laws used here involve fractality, diffusive transport, flux-volume proportionality, spatiotemporal energy, and thermal energies in a fractal volume. The unified formalism to calculate size distributions based on the scale-free probability conjecture (Equation (1)) appears to be a sound method to obtain (macroscopic) physical scaling laws in (microscopic) SOC systems. We mention as a caveat, however, that careful treatment has to be applied to small number statistics, truncation biases, data undersampling, background subtraction, inadequate fitting ranges, and deviations from ideal power-law functions.

3.2. Power-law Slopes

Our unified method of implementing physical scaling laws in the calculation of size (or occurrence rate) distribution functions yields a power-law slope α_x for every SOC parameter x. Thus, we have a unique correspondence of a scaling law with the power-law slope α. Our results yield a power-law slope of $\alpha_{E_1} = (13/9) = 1.44$ for an SOC system with spatiotemporal avalanche energies, a slope of $\alpha_{E_2} = (7/3) = 2.33$ for the thermal energy in an SOC system with 2D geometry, and $\alpha_{E_3} = (9/5) = 1.80$ for the thermal energy in an SOC system with 3D geometry. We can discard the model with the fractal 2D geometry, but it explains why some researchers found relatively high values of $\alpha_E > 2$. So we are left with relatively low values of $\alpha_E < 2$ for realistic energy models, such as $\alpha_{E_1} \approx 1.44$ for spatiotemporal avalanches, or $\alpha_{E_3} \approx 1.80$ for 3D fractal avalanches. Although we obtain a well-defined value for the power-law slope α_E for each size distribution, we should keep in mind that the estimation of fractal dimensions has some uncertainties within the fractal domains, such as in the ranges of $1 \leqslant D_A \leqslant 2$ and $2 \leqslant D_V \leqslant 3$, respectively (Aschwanden & Aschwanden 2008a, 2008b). In principle, one can measure the values of the fractal dimensions D_A and D_V from the observed (fitted) power-law slopes α_A and α_V, i.e., $D_A = 2/(\alpha_A - 1)$ and $D_V = 2/(\alpha_V - 1)$ (Table 1).

3.3. Nanoflares and Coronal Heating

It was pointed out early on that power-law distributions $N(E) \propto E^{-\alpha}$ of energies, with a slope flatter than the critical value of $\alpha_E = 2$, imply that the energy integral diverges at the

Table 4
Observed Frequency Distributions of Thermal Energies E_3 Based on 3D Fractal Model with Isotropic Line-of-sight Depth $h = \sqrt{A}$

Power-law Slope of Fluence or Energy α_{E3}	Instrument Wavelength	Observed Phenomenon	References
1.55 ± 0.05	Yohkoh	Solar flares	Shimizu (1995)
1.90	SOHO/EIT 195	EUV transient	Berghmans et al. (1998)
1.73 ± 0.28	SOHO/EIT 195	EUV transient	Berghmans & Clette (1999)
2.05 ± 0.05	TRACE 171,195	nanoflares	Parnell & Jupp (2000)
1.57 ± 0.05	Yohkoh SXT/AlMg	nanoflares	Aschwanden & Parnell (2002)
1.41 ± 0.09	Yokhoh SXT/AlMg	nanoflares	Aschwanden & Parnell (2002)
1.81 ± 0.10	TRACE 195	nanoflares	Aschwanden & Parnell (2002)
1.70 ± 0.17	TRACE 195	nanoflares	Aschwanden & Parnell (2002)
1.86 ± 0.07	TRACE 171	nanoflares	Aschwanden & Parnell (2002)
2.06 ± 0.10	TRACE 171	nanoflares	Aschwanden & Parnell (2002)
1.66	SOHO/EIT 195	nanoflares	Uritsky et al. (2007)
1.79 ± 0.01	AIA/SDO 171 A	coronal brightenings	Joulin et al. (2016)
1.83 ± 0.01	AIA/SDO 193 A	coronal brightenings	Joulin et al. (2016)
1.88 ± 0.01	AIA/SDO 211 A	coronal brightenings	Joulin et al. (2016)
1.80 ± 0.01	IRIS	nanoflares	Nhalil et al. (2020)
2.07 ± 0.02	IRIS	nanoflares	Nhalil et al. (2020)
2.00 ± 0.20	AIA/SDO	flares	Kawai & Imada (2022)
Outliers:			
(2.15 ± 0.01)[a]	AIA/SDO 131 A	coronal brightenings	Joulin et al. (2016)
(2.53 ± 0.01)[a]	AIA/SDO 335 A	coronal brightenings	Joulin et al. (2016)
(2.28 ± 0.03)[b]	AIA/SDO	nanoflares	Purkhart & Veronig (2022)
1.80 ± 0.18[c]			Mean of 17 observations
1.80			Theoretical prediction

Notes.
[a] No large events are detected in the 131 Å and 335 Å high-temperature bands during the time of observations, which causes a steeper power-law slope (Joulin et al. 2016).
[b] High-energy events could have significant uncertainties since they may heavily depend on accurate event combinations between many pixels, one of the most challenging steps in the event detection algorithm (Purkhart & Veronig 2022).
[c] Bold font indicates theoretical values, while roman font indicates observational values.

upper end $E_{\max}$, and thus the total energy of the distribution is dominated by the largest events (Hudson 1991),

$$E_{\rm tot} = \int_{E_{\min}}^{E_{\max}} E \times N(E) dE = \int_{E_{\min}}^{E_{\max}} (\alpha - 1) E^{1-\alpha_E} dE = \left(\frac{\alpha - 1}{2 - \alpha}\right) [E_{\max}^{2-\alpha} - E_{\min}^{2-\alpha_e}]. \quad (23)$$

In the opposite case, however, when the power-law distribution is steeper than the critical value, it will diverge at the lower end $E_{\min}$, and thus the total energy budget will be dominated by the smallest detected events, an argument that was used for dominant nanoflare heating (Krucker & Benz 1998). However, subsequent simulations demonstrated that there exists a strong bias toward a steeper slope ($\alpha_{E_2} \approx 2.3$–2.6) if the assumption of a constant line-of-sight depth is assumed ($h_0 = \text{const}$), while the application of an isotropic geometry ($h = A^{1/2}$) lowers the power-law slope to $\alpha \approx 2.0$ (Parnell & Jupp 2000; Benz & Krucker 2002). In our analytical 2D fractal model we predict a power-law slope of $\alpha_{E2} = (7/3) \approx 2.33$ (Table 3), which agrees well with the spread of observed values, $\alpha_{E_2} = 2.38 \pm 0.09$ (Table 3). This result shows clearly that the size distribution of nanoflares has a power-law slope of $\alpha < 2$, for both the spatiotemporal model ($\alpha_{E_1} = (13/9) \approx 1.44$), as well as for the 3D fractal thermal energy model ($\alpha_{E_3} = (9/5) = 1.80$), which implies that the energy in nanoflares does not diverge at the lower end, $E_{\min} \lesssim 10^{24}$ erg, and that nanoflares are not the dominant contributor to the heating of the solar corona.

Alternatively, one could argue that the magnetic reconnection process, considered to be the primary mechanism of nanoflares, occurs in quasi-2D current sheets, which would naturally lead to $V \propto A$. Averaging over many randomly oriented current sheets would give the same results, assuming that their orientation is independent of the size, which seems to be the case in MHD turbulence. Sub-MHD-scale simulations demonstrate that the reconnecting current sheets can actually break into quasi-1D current filaments for which the $V \propto A$ scaling would also hold, and both V and A would scale linearly with L, probably yielding an even steeper energy distribution slope (Daughton et al. 2011). This way the conditions for $\alpha > 2$ can in fact be possible. The SOC nanoflare scenario may exist although the conditions can be nontrivial. The presented analysis shows that the nanoflare heating in an SOC-like coronal plasma environment requires a particular magnetic geometry of the reconnecting plasma region. If this is the case, this could be an important new constraint for future simulations and observations.

4. Conclusions

In this study we test whether the standard self-organized criticality model can predict the size (or occurrence frequency) distribution functions $N(x)dx \propto x^{-\alpha_x}$ of physical parameters x in solar flares, down to the nanoflare regime with energies of

$E \gtrsim 10^{24}$ erg. We focus mostly on energy parameters, such as the spatiotemporal avalanche energy (E_1), the 2D fractal energy model (E_2), and the more realistic 3D fractal energy model (E_3). For this three energy models, power-law slopes of $\alpha_{E1} = 1.44$, $\alpha_{E2} = 2.33$, and $\alpha_{E3} = 1.80$ are predicted. We test these predictions from literature values and find mean slopes of $\alpha_{E1} = 1.47 \pm 0.07$ from 9 data sets (Table 2), $\alpha_{E2} = 2.38 \pm 0.09$ from 4 data sets (Table 3), and $\alpha_{E3} = 1.80 \pm 0.18$ from 17 data sets (Table 4), which all are fully self-consistent with the predicted values.

The related observations include solar flares observed in hard X-rays, soft X-rays, and EUV wavelengths, from large flares with energies of $E \lesssim 10^{33}$ erg down to nanoflares (specified as EUV transients, coronal brightenings, or blinkers). We consider both the spatiotemporal (or standard SOC) model as well as the 3D fractal energy model, based on emission measure analysis, as realistic tools to quantify the energy of flares and nanoflares, while the 2D version of the fractal energy model (E_2) significantly overestimates the power-law slope of the energy size distributions. The analytical approach clearly demonstrates that the size distribution of nanoflares has a power-law slope of $\alpha < 2$, and thus the energy in nanoflares does not diverge at the lower end of the size distributions, so that nanoflares do not qualify to be dominant contributors to the heating of the solar corona.

While numerical Monte Carlo–type simulations leave the option of a supercritical value of $\alpha_E \gtrsim 2$ open (Krucker & Benz 1998; Parnell & Jupp 2000), we demonstrate in this Letter that this conclusion is true only for the unrealistic 2D fractal energy model E_2, observationally ($\alpha_{E_2} = 2.38 \pm 0.09$), as well as theoretically ($\alpha_{E_2} = (7/3) \approx 2.33$). Consequently, the power-law slope is flatter ($\alpha_E < 2$) for at least two energy models (the spatiotemporal standard SOC model $\alpha_{E_1} = 1.44$, and the 3D fractal energy model $\alpha_{E_3} = 1.80$), which implies that heating of the corona in active regions is dominated by large (M- and X-class) flares. The same argument holds for quiet-Sun regions, where the largest events in each size distribution (of nanoflares, microflares, EUV transients, coronal brightenings, blinkers, etc.) dominate the energy budget (see power-law slopes α_{E_3} of energies in Table 4), rather than the smallest events.

Appendix
SOC Glossary

In this appendix we provide a glossary of the most basic terms used in the SOC models described in this Letter.

area A: a two-dimensional spatial measure that scales with $A \propto L^2$ as a function of a length scale L (in the case of nonfractal geometry). A fractal area can be extracted from an image $I(X, Y)$ by identifying the sum of all image pixels above some threshold I_0, i.e., $I(X, Y) \geqslant I_0$.

avalanche event: the coherent spatiotemporal evolution $I(X, Y, t)$ of a randomly triggered instability.

cellular automaton: the original Bak–Tang–Wiesenfeld SOC model, which results from numerically simulated next-neighbor interactions in a lattice grid geometry.

energy E: of an avalanche event is the integral of the time-dependent flux $F(t)$ or intensity $I(t)$, integrated over the thresholded area $A(t)$, often approximated with the product of the flux $F(t)$ and event duration T, i.e., $E \approx F \times T$.

Euclidean dimension D: characterizing the spatial topology of an SOC event, such as a curvilinear geometry L (with Euclidean dimension $D = 1$), an area A (with Euclidean dimension $D = 2$), or a volume V (with Euclidean dimension $D = 3$).

emission measure: EM is closely related to the flux F or intensity I emitted by free–free bremsstrahlung in the optically thin regime, defined by the product of the squared electron density n_e^2 and volume V, i.e., $EM = n_e^2 V$.

event duration: measured from the time interval T where the flux or intensity $I(t) > I_0$ exceeds a threshold value I_0, attributed to an event-unrelated background.

fluence: F is the time integral E of the flux $F(t)$, i.e., $E = \int F(t)\, dt$, often approximated with the product of the flux maximum and the event duration, $E \approx max[F(t)] \times T$.

fractal dimension D: the geometric scaling of an area A with the length scale L, i.e., $D_A = \log(A)/\log(L)$, or of the volume, i.e., $D_V = \log(V)/\log(L)$. It is an extension of the Euclidean dimension (with integer values $D = 1, 2, 3$), to noninteger values.

frequency occurrence distribution function: identical to the term "size distribution," which for power-law functions is defined as $N(x)dx \propto x^{-\alpha_x}\, dx$, with α_x being the power-law index, generally being different for every physical parameter x.

Hausdorff dimension: equivalent to fractal dimension.

lattice grid: a numerical 3D Cartesian coordinate system [X_i, Y_j, Z_k] (in the case of three dimensions), which is used to track next-neighbor interactions in cellular automaton simulations.

nonfractal: equivalent to Euclidean dimensions $D = 1, 2, 3$.

nonlinear growth: the spatiotemporal evolution of an SOC avalanche after it has been triggered by a random disturbance of next-neighbor interactions.

pixel: the spatial element of a two-dimensional (2D) lattice grid.

power law: a generic function $N(x)dx = N_0 x^{-\alpha} dx$ that results from initial nonlinear growth and saturation after a random timescale (Rosner & Vaiana 1978).

size distribution: equivalent to occurrence frequency distribution function.

scale-free probability conjecture: the relationship $N(L)dL \propto L^{-D}$, expressing the reciprocal relationship between the number of events $N(L)$ and the size L of the events, where D is the fractal (or nonfractal) dimensionality (Aschwanden 2012).

scaling law: describing the scale invariance found in many nonlinear evolution processes. The functional relationship is of the form of power indices, e.g., $z(X, Y) \propto X^{\alpha} Y^{\beta}$.

spatiotemporal event: including both transport in space and time, such as the standard diffusion equation, i.e., $L(T) \propto T^{1/2}$.

self-organized criticality (SOC): the generic term of the Bak–Tang–Wiesenfeld model (Bak et al. 1987, 1988).

volume: V is a three-dimensional volume that scales with $V \propto L^3$ (in the case of a nonfractal geometry).

voxel: the spatial element of a discretized three-dimensional (3D) lattice grid.

waiting time distribution: the distribution of time intervals $\Delta t_i = t_{i+1} - t_i$ that are measured between subsequent SOC avalanches.

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