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The Domination Number of $P_m \times \overrightarrow{P_n}$

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

A mixed graph $G_M(V, E, A)$ is a graph containing unoriented edges (set E) as well as oriented edges (set A), referred to as arcs. In this paper we calculate the domination number of the Cartesian product of a path P_m with directed path $\overrightarrow{P_n}$ (mixed-grid graph ($P_m \times \overrightarrow{P_n}$) for $8 \leq m \leq 10$ and arbitrary n.

Keywords: Graph; directed graph; Cartesian product; path; directed path; mixed graph; mixed-grid graph; dominating set; domination number.

2010 Mathematics subject classification: 05C69, 05C38.

1 Introduction

All graphs and digraphs are assumed to be loopless and without duplicate edges or arcs. A mixed graph GM(V, E, A) is a graph containing unoriented edges (set E) as well as oriented edges (set A), referred to as arcs. This notion was first introduced in [1].

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Let $G = (V_1, E)$ be a graph and $D = (V_2, A)$ be a digraph. The Cartesian product $G \times D$ is the *mixed-graph* with vertex set $V(G \times D) = V_1(G) \times V_2(D)$ and edge (arc) set is $((u_1,v_1),(u_2,v_2)) \in E(G \times D)$ if and only if either $v_1 = v_2$ and $(u_1, u_2) \in E(G)$ or $u_1 = u_2$ and $(v_1, v_2) \in A(D)$. A subset S of the vertex set $V(G \times D)$ is a dominating set of $G \times D$ if for each vertex $v \in G \times D$ there exists a vertex $u \in S$ such that (u, v) is an edge (arc) of $G \times D$. The domination number of $G \times D$, $\gamma(G \times D)$, is the cardinality of the smallest dominating set of $G \times D$.

Let P_m be a path with vertex set $V(P_m) = \{1, 2,..., m\}$, and edge set $E(P_m) = \{(i, i+1) : 1 \le i \le m-1\}$, and let $\overrightarrow{P_n}$ be a directed path with vertex set $V(P_n) = \{1, 2, \dots, n\}$, and arc set $A(P_n) = \{(i, i+1) : 1 \le i \le n-1\}$. Then for Cartesian product P_m and $\overrightarrow{P_n}$ is mixed-grid graph $P_m \times \overrightarrow{P_n}$ with $V(P_m \times \overrightarrow{P_n}) = \{(i, j): 1 \le i \le m, 1 \le j \le n\}$, such that there is an arc from (i, j) to (p, q) if and only if i = p and q- j =1 and is an edge from (i, j) to (p, q) if and only if j = q and p - i =1. The i*th* row of $V(P_m \times \overrightarrow{P_n})$ is R_i = {is a directed path $\overrightarrow{P_n}$ (i, j) : j =1, 2,..., n}, and the j*th* column K_j ={(i, j) : is a path P_m (i, j) : i =1, 2, ..., m}. If S is a dominating set for $P_m \times \overrightarrow{P_n}$, then we denote $W_j = S \cap K_j$. Let $s_j = |W_j|$, where the sequence $(s_1, s_2, ..., s_n)$ is called a dominating sequence corresponding to S. For $1 \le j \le n$, the vertices of *j*-column are dominated by vertices of *j*-column or (*j*-1)-column. The vertices of the first column are dominated only by vertices of K_1. Also, for $1 \le i \le m$, the vertices of *i*-row are dominated by vertices of *i*-row, (*i* -l)-row or (*i* +l)-row. The vertices of the first row are dominated only by vertices of R_1 or R_2. Thus the following is true:

$$\gamma(P_m \times P_n) \le \gamma(P_m \times \overrightarrow{P_n}) \le \gamma(P_m \times \overrightarrow{P_n}).$$

For finding domination number of grid graphs $P_m \times P_n$, Jacobson and Kinch in [2], were calculated the domination number of cartesian product of undirected paths P_m and P_n for m = l, 2, 3, 4. The cases m = 5, 6 were calculated by Chang and Clark [3]. Also, Chang et al. [4], established the upper bounds of cartesian product of undirected paths P_m and P_n for $5 \le m \le 10$ and arbitrary n. In [5], Gravier and Mollard given un upper and lower bounds of general cartesian product of two undirected paths. Goncalves et al., [6] proved Chang's conjecture saying that for every $16 \le n \le m$, $\gamma(P_m \times P_n) = \lfloor (n+2)(m+2)/5 \rfloor - 4$.

For domination number of directed grid graphs, Liu et al. [7], they studied the domination number of $\overrightarrow{P_m} \times \overrightarrow{P_n}$ for m = 2, 3, 4, 5, 6 and arbitrary n. Also, in [8] the author studied the domination number of $\overrightarrow{P_m} \times \overrightarrow{P_n}$ for arbitraries m and n. Also, in [9] we have the following results:

$$\gamma(P_1 \times \overrightarrow{P_n}) = \gamma(\overrightarrow{P_n}) = \left\lceil \frac{n}{2} \right\rceil.\ \gamma(P_2 \times \overrightarrow{P_n}) = n.\ \gamma(P_3 \times \overrightarrow{P_n}) = n.$$

$$\gamma(P_4 \times \overrightarrow{P_n}) = \left\lceil \frac{3n}{2} \right\rceil.\ \gamma(P_5 \times \overrightarrow{P_n}) = \left\lceil \frac{3n}{2} \right\rceil + 1.\ \gamma(P_6 \times \overrightarrow{P_n}) = 2n.$$

$$\gamma(P_7 \times \overrightarrow{P_n}) = 2n + 2.$$

2 Main Results

In this section we calculate the domination number of the Cartesian product of a path $P_m$and a directed path $\overrightarrow{P_n}$ for m = 8, 9, 10 and arbitrary n.

Observation 2.1. Since for each vertex (i, j)$\in$V($P_m \times \overrightarrow{P_n}$) has two undirected degrees in V(K_j), one outdegree in V(K_{j+1}) and one indegree from V(K_{j-1}), then can it dominates at must four vertices of $P_m \times \overrightarrow{P_n}$ with itself. Thus implies that $\gamma(P_m \times \overrightarrow{P_n}) \geq mn / 4$. □

Observation 2.2. Let S be a dominating set of $P_m \times \overrightarrow{P_n}$. Since the vertices of the first column are dominated only by vertices of K_1. Also, for $2 \leq j \leq n$, the vertices of j-column are dominated by vertices of j-column or (j-1)-column. Then the following are holds:

- **i.** $s_1 \geq \lceil m/3 \rceil$.
- **ii.** $s_j + 3s_{j+1} \geq m$ for all j = 1, …, n. □

Lemma 2.1. There is a minimum dominating set S for $P_m \times \overrightarrow{P_n}$ with dominating sequence (s_1, s_2, ..., s_n) such that for all j = 1, 2, ..., n, is $\lceil m/3 \rceil \leq s_j \leq \lceil m/2 \rceil$, where m > 1.

Proof. Let S be a minimum dominating set for $P_m \times \overrightarrow{P_n}$ with dominating sequence (s_1, s_2, ..., s_n). Assume that for some j, s_j is large. Then we modify S by moving some vertices from column j to column j+1, such that the resulting set is still dominating set for $P_m \times \overrightarrow{P_n}$ (because each vertex in S$\cap K_j$ is dominates only vertices from K_j and K_{j+1}). For $1\leq i \leq m$ and $1\leq j \leq n$, let W = S$\cap${(i, j), (i +1, j), (i+2, j), (i+3, j)}. If |W| ≥ 3, then we have three cases:

Case 1. If {(1, j), (2, j)}$\subseteq$S or {(m-1, j), (m, j)}$\subseteq$S. Then we can move (1, j) to (1, j +1) or (m, j) to (m, j +1). Furthermore, S is still dominating set of $P_m \times \overrightarrow{P_n}$.

Case 2. |W| = 4, then we put S_1= (S-W)$\cup${(i, j), (i +1, j +1), (i+2, j +1), (i+3, j)}.

Case 3. |W| = 3, then we have two sub cases:

SubCase 3.1. |W| = 3 and W = S$\cap${(i, j), (i +1, j), (i+2, j), (i+3, j)}= {(i, j), (i +1, j), (i+2, j)} or W = (i +1, j), (i+2, j), (i+3, j)}. Two cases are similar by symmetry, for the first we put S_1= (S-W)$\cup${(i, j), (i +2, j), (i +1, j +1)} and for the second we put S_1= (S-W)$\cup${(i +1, j), (i +3, j), (i+2, j+1)}.

SubCase 3.2. |W| = 3 and W = S$\cap${(i, j), (i+1, j), (i+2, j), (i+3, j)}= {(i, j), (i+1, j), (i+3, j)} or W = {(i, j), (i+2, j), (i+3, j)}. Also, two cases are similar by symmetry. Then we change S, respectively as follows: S_1 = (S-W)$\cup${(i, j), (i+3, j), (i+1, j +1)}, S_1= (S-W)$\cup${(i, j), (i+3, j), (i+2, j +1)}, see Fig. 1 for cases 2, 3. We repeat this process if necessary eventually leads to a dominating set with required properties. Also, we get S_1 is a dominating set for $P_m \times \overrightarrow{P_n}$ with |S| = |S_1|. Thus, we can assume that every four consecutive vertices of the j'th column include at most two vertices of S. This implies that $s_j \leq \lceil m/2 \rceil$, for all $1 \leq j \leq n$.

To prove the lower bound, we suppose that |$K_{j-1}\cap$D| is be a maximum, i.e., $s_{j-1} = \lceil m/2 \rceil$. Then for each five vertices in K_j must include at last one vertex from S, otherwise K_j contain three successive vertices from S. This contradiction with the upper bounds. Thus we get $s_j \geq \lceil m/5 \rceil$ for all $1 \leq j \leq n$. □

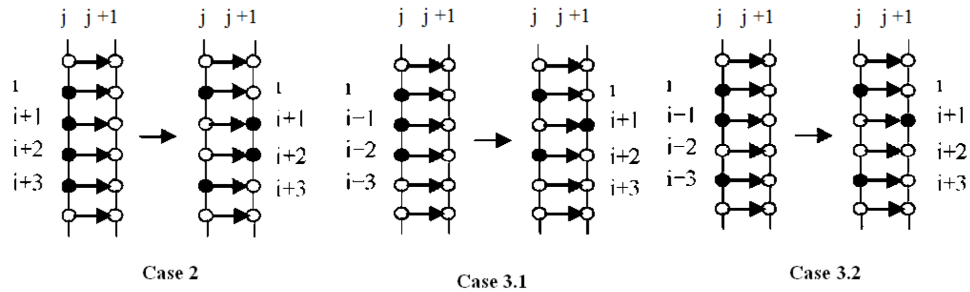

Fig. 1. Modify S

By Lemma 2.1, always we have a minimum dominating set S with dominating sequence $(s_1, s_2,...,s_n)$, such that $\lceil m/5 \rceil \le s_j \le = \lceil m/2 \rceil$, for all j = 1, 2, …, n. So, for all the next, we consider a minimum dominating set of $P_m \times \overrightarrow{P_n}$ with dominating sequence $(s_1, …, s_n)$ with $\lceil m/5 \rceil \le s_j \le \lceil m/2 \rceil$, for all j = 1, 2, ..., n.

Proposition 2.1.

- **i.** $\gamma(P_m \times \overrightarrow{P_n}) \ge \gamma(P_r \times \overrightarrow{P_n}) + \gamma(P_{m-r-2} \times \overrightarrow{P_n})$.
- **ii.** $\gamma(P_m \times \overrightarrow{P_n}) \le \gamma(P_r \times \overrightarrow{P_n}) + \gamma(P_{m-r} \times \overrightarrow{P_n})$.

Proof. For i and ii, the proofs are easy. □

Proposition 2.2. $\gamma(P_8 \times \overrightarrow{P_n}) \le \left\lceil \frac{5n}{2} \right\rceil + 1$.

Proof. Let S be a set defined as follows:

$$S = \{(1, 1), (3, 1), (5, 1), (7, 1)\} \cup \{(3, 2j), (7, 2j) : 1 \le j \le \lfloor n/2 \rfloor\} \cup \{(1, 2j+1), (5, 2j+1), (8, 2j+1) : 1 \le j \le \lfloor (n-1)/2 \rfloor\}.$$

We have $|S| = 4 + 2\lfloor n/2 \rfloor + 3\lfloor (n-1)/2 \rfloor$, also S is a dominating set of $P_8 \times \overrightarrow{P_n}$ (see Fig. 2, for $\gamma(P_8 \times \overrightarrow{P_{11}})$). Thus $\gamma(P_8 \times \overrightarrow{P_n}) \le 4 + 2\left\lfloor \frac{n}{2} \right\rfloor + 3\left\lfloor \frac{n-1}{2} \right\rfloor = \left\lceil \frac{5n}{2} \right\rceil + 1$. □

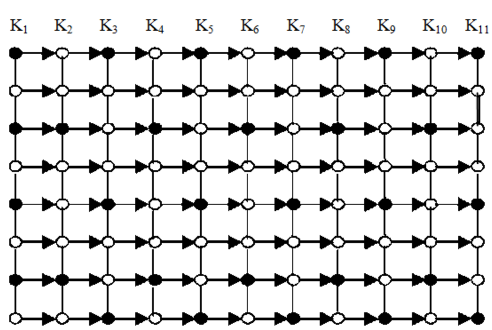

Fig. 2. A dominating set of $P_8 \times \overrightarrow{P_{11}}$

Proposition 2.3.

- **i.** The case $(s_1, s_2, s_3) = (3, 3, 2)$ is not possible.
- **ii.** The case $(s_1, s_2, s_3) = (4, 2, 2)$ is not possible.
- **iii.** There are four possibilities for $(s_j, s_{j+1}) = (2, 2)$.
- **iv.** The case $(s_j, s_{j+1}, s_{j+2}) = (2, 2, 2)$ is not possible.
- **v.** The case $(s_j, s_{j+1}, s_{j+2}, s_{j+3}) = (2, 2, 3, 2)$ is not possible.

Proof. i and ii by drawing.

iii. For $(s_j, s_{j+1}) = (2, 2)$ we have four cases are:

a. $\{(1, j), (2, j), (4, j+1), (7, j+1)\}$, **b.** $\{(1, j), (5, j), (3, j+1), (7, j+1)\}$,
c. $\{(1, j), (8, j), (3, j+1), (6, j+1)\}$, **d.** $\{(4, j), (5, j), (2, j+1), (7, j+1)\}$,

iv. For all cases in iii, we get $(s_j, s_{j+1}, s_{j+2}) = (2, 2, 2)$ is not possible.

v. By Completed drawing of cases $(s_j, s_{j+1}) = (2, 2)$, we deduce that $(s_j, s_{j+1}, s_{j+2}, s_{j+3}) = (2,2,3,2)$ is not possible.

Proposition 2.4.

i. $s_1 + s_2 \geq 6$.
ii. $s_1 + s_2 + s_3 \geq 9$.
iii. $s_1 + s_2 + s_3 + s_4 \geq 11$.
iv. $s_1 + s_2 + s_3 + s_4 + s_5 \geq 14$.

Proof. **i.** We have $s_1 \geq 3$ and $2 \leq s_j \leq 4$. If $s_1 = 4$, then $s_2 \geq 2$ and so $s_1 + s_2 \geq 6$. Let $s_1 = 3$, then needs $|\{(1,1), (2,1), (7,1), (8,1)\} \cap S| = 2$, at the same time $\{(1,1), (2,1), (7,1), (8,1)\} \cap S \neq \{(1, 1), (8, 1)\}$. Suppose $\{(1, 1), (2, 1), (7, 1), (8, 1)\} \cap S = \{(1, 1), (7, 1)\}$ or $\{(2, 1), (7, 1)\}$ where two cases $\{(1, 1), (2, 1), (7, 1), (8, 1)\} \cap S = \{(1, 1), (7, 1)\}$ or $\{(2, 1), (8, 1)\}$ are similar by symmetry. If $\{(1, 1), (2, 1), (7, 1), (8, 1)\} \cap S = \{(1, 1), (7, 1)\}$, then we need $(4, 1) \in S$ and $s_2 \geq 3$. Thus we get $s_1 + s_2 \geq 6$. Let $\{(1, 1), (2, 1), (7, 1), (8, 1)\} \cap S = \{(2, 1), (7, 1)\}$, then $(4, 1) \in S$ or $(5, 1) \in S$, the two cases are similar by symmetry. Assume that $(5, 1) \in S$, then $s_2 \geq 3$. Thus we get $s_1 + s_2 \geq 6$.

ii. From i we have $s_1 + s_2 \geq 6$. If $s_1 + s_2 \geq 7$, then $s_1 + s_2 + s_3 \geq 9$ {because $s_j \geq 2$ for all $j = 1, \ldots, n$}. Suppose $s_1 + s_2 = 6$, then immediately from Proposition 2.3 (i and ii).

iii. It clear from ii and $s_j \geq 2$ for all $j = 1, \ldots, n$.

iv. From iii, $s_1 + s_2 + s_3 \geq 9$. If $s_1 + s_2 + s_3 \geq 10$, then finish {because $s_j \geq 2$}. Let $s_1 + s_2 + s_3 = 9$ and suppose that $\sum_{j=1}^{5} s_j < 14$. Then we must have $s_4 = s_5 = 2$. By proposition 2.3 (iii), we have four cases for $(s_4, s_5) = (2, 2)$. But for all cases we Get $s_1 + s_2 + s_3 \geq 10$, and this a contradiction. Finally we get $\sum_{j=1}^{5} s_j \geq 14$. □

Theorem 2.1. $\gamma(P_8 \times \overrightarrow{P_n}) = \left\lceil \frac{5n}{2} \right\rceil + 1$.

Proof. By Lemma 2.1, $2 \le s_j \le 4$. Observation 2.2, gets $s_1 \ge 3$ and $s_j + 3s_{j+1} \ge m$. Which implies that, if $s_j = 2$ or 3 is $s_{j+1} \ge 2$. By Proposition 2.3, for each four columns including 10 vertices from S. We consider four cases:

Case a. $n \equiv 0 (\mathrm{mod}\ 4)$. By Proposition 2.4, $s_1 + s_2 + s_3 \ge 9$ and $s_1 + s_2 + s_3 + s_4 \ge 11$ {because $s_j \ge 2$ for all j = 1, …, n}. Thus

$$\gamma(P_8 \times \overrightarrow{P_n}) = \sum_{j=1}^{n} s_j \ge \sum_{j=1}^{4} s_j + \sum_{j=5}^{n} s_j \ge 11 + 10\,\frac{n-4}{4} = \frac{5n+2}{2}.$$

Case b. $n \equiv 1 (\mathrm{mod}\ 4)$. By Proposition 2.4, $\sum_{j=1}^{5} s_j \ge 14$. Then

$$\gamma(P_8 \times \overrightarrow{P_n}) = \sum_{j=1}^{n} s_j \ge \sum_{j=1}^{5} s_j + \sum_{j=6}^{n} s_j \ge 14 + 10\,\frac{n-5}{4} = \frac{5n+3}{2}.$$

Case c. $n \equiv 2 (\mathrm{mod}\ 4)$. By Proposition 2.4, $s_1 + s_2 \ge 6$. Also, gets

$$\gamma(P_8 \times \overrightarrow{P_n}) = \sum_{j=1}^{n} s_j \ge \sum_{j=1}^{2} s_j + \sum_{j=3}^{n} s_j \ge 6 + 10\,\frac{n-2}{4} = \frac{5n+2}{2}.$$

Case d. $n \equiv 3 (\mathrm{mod}\ 4)$. By Proposition 2.4, $s_1 + s_2 + s_3 \ge 9$. So

$$\gamma(P_8 \times \overrightarrow{P_n}) = \sum_{j=1}^{n} s_j \ge \sum_{j=1}^{3} s_j + \sum_{j=4}^{n} s_j \ge 9 + 10\,\frac{n-3}{4} = \frac{5n+3}{2}.$$

For all the cases, we get

$$\gamma(P_8 \times \overrightarrow{P_n}) \ge \left\lceil \frac{5n}{2} \right\rceil + 1.$$

Finally, Proposition 2.2 together with the last result gets

$$\gamma(P_8 \times \overrightarrow{P_n}) = \left\lceil \frac{5n}{2} \right\rceil + 1. \qquad \square$$

Proposition 2.5.

$$\gamma(P_9 \times \overrightarrow{P_n}) \le 3n : n \le 3.$$

$$\gamma(P_9 \times \overrightarrow{P_n}) \le \left\lceil \frac{5n}{2} \right\rceil + 2 : n \ge 4.$$

Proof. For $n \ge 3$, let $S_1 = \{(2, j), (5, j), (8, j)$ for $j = 1, \ldots, n\}$. If $n \ge 4$, we define S_2 as follows:

$$S_2 = \{(3,1), (7,1)\} \cup \{(1, 2j-1), (5, 2j-1), (9, 2j-1): 1 \le j \le \lceil n/2 \rceil\} \cup \{(3, 2j), (7, 2j): 1 \le j \le \lfloor n/2 \rfloor\}.$$

The set S_1 is a dominating set of $P_9 \times \overrightarrow{P_n}$ for n ≤ 3 with $|S_1| = 3n$.

Also, S_2 is a dominating set of $P_9 \times \overrightarrow{P_n}$ for n ≥4 with $|S_2| = 2 + 3\lceil n/2 \rceil + 2\lfloor n/2 \rfloor = \lceil 5n/2 \rceil + 2$. (see Fig. 3, for $\gamma(P_9 \times \overrightarrow{P_{13}})$). Thus

$$\gamma(P_9 \times \overrightarrow{P_n}) \leq 3n : n \leq 3. \tag{1}$$

$$\gamma(P_9 \times \overrightarrow{P_n}) \leq \left\lceil \frac{5n}{2} \right\rceil + 2 : n \geq 4. \tag{2}$$ □

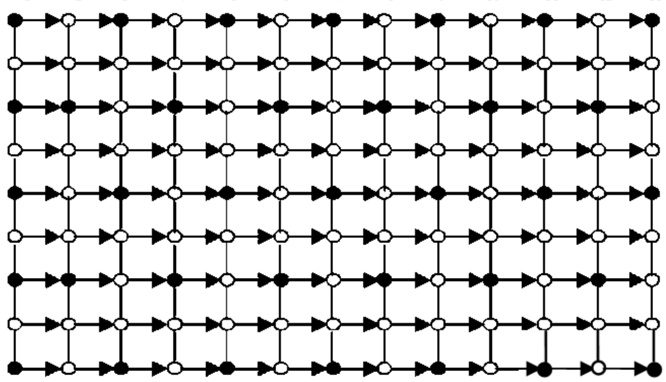

Fig. 3. A dominating set of $P_9 \times P_{13}$

Proposition 2.6.

i. If $s_2 = 2$ then $s_1 = 5$.
ii. If s3 = 2 then $s_1 + s_2 \geq 8$.
iii. If $(s_1, s_2, \ldots, s_j) = (3, 3, \ldots, 3)$ then $s_{j+1} \geq 3$ for j ≥ 2.
iv. If K_j the first column, such that $s_j = |K_j \cap S| = 2$, then $\sum_{d=1}^{j-1} s_d \geq 3(j-1)+1$ for j >1.

Proof. **i** and **ii,** immediately by drawing.

iii. There is one possible for $s_1 = 3$ is $K_1 \cap S = \{(2, 1), (5, 1), (8, 1)\}$. This implies that $s_2 \geq 3$. If $s_2 = 3$ then $K_2 \cap S = \{(2, 2), (5, 3), (8, 3)\}$. Furthermore, if $s_1 = s_2 = \ldots = s_j = 3$, then $K_j \cap S = \{(2, j), (5, j), (8, j)\}$. Thus gets $s_{j+1} \geq 3$.

iv. Immediately from iii. □

Theorem 2.2.

$$\gamma(P_9 \times \overrightarrow{P_n}) = 3n : n \leq 3.$$

$$\gamma(P_9 \times \overrightarrow{P_n}) = \left\lceil \frac{5n}{2} \right\rceil + 2 : n \geq 4.$$

Proof. From Observation 2.2 and Lemma 2.1, we have $s_1 \geq 3$ and $2 \leq s_j \leq 5$. Furthermore, if $s_j = 2$ then $s_{j.1} \geq 3$, i.e., $(s_j, s_{j+1}) = (2, 2)$ is not possible. This implies that, if $s_j = 2$ then

$\sum_{d=j}^{j+r} s_d \geq \left\lfloor \frac{5(r+1)}{2} \right\rfloor$ for $r \geq 1$. We consider two cases:

Case a. If $S_J \geq 3$ for J = 1, …, n, then $\gamma(P_9 \times \overrightarrow{P_n}) = \sum_{j=1}^{n} s_j \geq 3n$ for $n \leq 3$. Also for $n \geq 4$ is $\gamma(P_9 \times \overrightarrow{P_n}) \geq 3n \geq \left\lceil \frac{5n}{2} \right\rceil + 2$. Then, (1) and (2) together with last results, gets the required.

Case b. $s_j = 2$ for some $j \geq 2$. Suppose K_j is the first column, such that $s_j = 2$. We consider the following subcases:

SubCase b.1. j = 2. By Proposition 2.6, $s_1 = 5$ and $s_3 \geq 3$. Then $\gamma(P_9 \times \overrightarrow{P_n}) = \sum_{j=1}^{n} s_j \geq 3n$ where $n \leq 3$. For $n \geq 4$, by Proposition 2.6(iv), we have

$$\gamma(P_9 \times \overrightarrow{P_n}) = \sum_{j=1}^{n} s_j = s_1 + \sum_{j=2}^{n} s_j \geq 5 + \left\lfloor \frac{5(n-1)}{2} \right\rfloor = \left\lceil \frac{5n}{2} \right\rceil + 2.$$

SubCase b.2. j = 3. By Proposition 2.6, $s_1 + s_2 \geq 8$. Thus $\gamma(P_9 \times \overrightarrow{P_n}) = \sum_{j=1}^{n} s_j \geq 3n$ where $n \leq 3$. For $n \geq 4$, Proposition 2.6(iv), implies

$$\gamma(P_9 \times \overrightarrow{P_n}) = \sum_{j=1}^{n} s_j = s_1 + s_2 + \sum_{j=2}^{n} s_j \geq 8 + \left\lfloor \frac{5(n-2)}{2} \right\rfloor \geq \left\lceil \frac{5n}{2} \right\rceil + 2.$$

SubCase b.3. $j \geq 4$. From Proposition 2.6(iv), we get

$$\gamma(P_9 \times \overrightarrow{P_n}) = \sum_{j=1}^{n} s_j = \sum_{d=1}^{j-1} s_d + \sum_{d=j}^{n} s_d \geq 3(j-1) + 1 + \left\lfloor \frac{5(n-j+1)}{2} \right\rfloor$$

{because $\gamma(P_9 \times \overrightarrow{P_n})$ is natural number}, then $\gamma(P_9 \times \overrightarrow{P_n}) \geq \left\lceil \frac{5n}{2} \right\rceil + 2$. Thus, for all cases, we have

$$\gamma(P_9 \times \overrightarrow{P_n}) \geq \left\lceil \frac{5n}{2} \right\rceil + 2.$$

The last result together with (2), gets

$$\gamma(P_9 \times \overrightarrow{P_n}) = \left\lceil \frac{5n}{2} \right\rceil + 2. \qquad \square$$

Proposition 2.7.

$$\gamma(P_{10} \times \overrightarrow{P_n}) \le 3n + 1 : n \equiv 0 (\mathrm{mod} \;\; 2).$$

$$\gamma(P_{10} \times \overrightarrow{P_n}) \le 3n + 2 : n \equiv 1 (\mathrm{mod} \;\; 2).$$

Proof. Let S be a set defined as follows:

S = {(1,1), (4,1), (7,1), (10,1), (3,2), (5,2), (8,2)}
∪{(1, 2j+1), (5, 2j+1), (6, 2j+1), (10, 2j+1) : 1≤ j ≤⌈n/2⌉-1}
∪{(3, 2j), (8, 2j) : 2 ≤ j ≤⌊n/2⌋}.

S is a dominating set of $P_{10} \times \overrightarrow{P_n}$ with |S| = 7 + 4(⌈n/2⌉-1) + 2(⌊n/2⌋) = 3n + 1 for n is even and |S| = 3n + 2 for n is odd (see Fig. 4, for $\gamma(P_{10} \times \overrightarrow{P_n})$). Thus

$$\gamma(P_{10} \times \overrightarrow{P_n}) \le 3n + 1 : n \equiv 0 (\mathrm{mod} \;\; 2).$$
$$\gamma(P_{10} \times \overrightarrow{P_n}) \le 3n + 2 : n \equiv 1 (\mathrm{mod} \;\; 2). \qquad \square$$

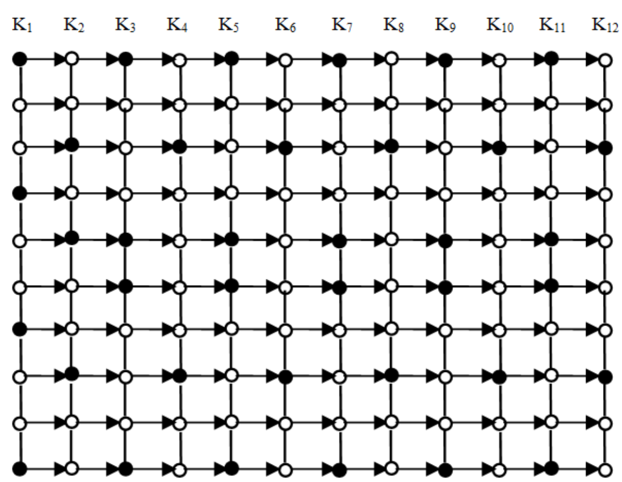

Fig. 4. A dominating set of $P_{10} \times P_{12}$

Proposition 2.8.

i. If $s_2 = 2$ then $s_1 = 6$.
ii. If $s_2 = 3$ then $s_1 \ge 4$.

Proof. i and ii, immediately by drawing.

Theorem 2.3.

$$\gamma(P_{10} \times \overrightarrow{P_n}) = 3n + 1 : n \equiv 0 (\text{mod } 2).$$
$$3n + 1 \leq \gamma(P_{10} \times \overrightarrow{P_n}) \leq 3n + 2 : n \equiv 1(\text{mod } 2).$$

Proof. By Observation 2.2 and Lemma 2.1, we have $s_1 \geq 4$ and $2 \leq s_j \leq 5$. Furthermore, $(s_j, s_{j+1}) = (3, 2)$ is not possible {Observation 2.2(ii)}. Then by Proposition 2.8, gets $\gamma(P_{10} \times \overrightarrow{P_n}) \geq 3n + 1$. Proposition 2.7, together with the last result, produces,

$$\gamma(P_{10} \times \overrightarrow{P_n}) = 3n + 1 : n \equiv 0 (\text{mod } 2).$$
$$3n + 1 \leq \gamma(P_{10} \times \overrightarrow{P_n}) \leq 3n + 2 : n \equiv 1(\text{mod } 2).$$

In another way, by Proposition 2.1 we get $\gamma(P_{10} \times \overrightarrow{P_n}) \geq 2\gamma(P_4 \times \overrightarrow{P_n}) = 2\left\lceil \frac{3n}{3} \right\rceil$.

Thus

$$\gamma(P_{10} \times \overrightarrow{P_n}) \geq 2\gamma(P_4 \times \overrightarrow{P_n}) = 2\left\lceil \frac{3n}{3} \right\rceil = 3n + 1, \textit{where} \quad n \equiv 1(\text{mod } 2).$$

Then Proposition 2.7, including

$$3n + 1 \leq \gamma(P_{10} \times \overrightarrow{P_n}) \leq 3n + 2 : n \equiv 1(\text{mod } 2).$$ □

3 Conclusion

In this paper, we find the domination numbers of the Mixed-graph $P_m \times \overrightarrow{P_n}$ for m = 8, 9, 10 and arbitrary n. As a future work, we would like to work on the bounds of $\gamma(P_{10} \times \overrightarrow{P_n})$ for arbitraries m and n.

Competing Interests

Authors have declared that no competing interests exist.

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